A total of 60 points is distributed among the following problems

1. [30 pts, 5pts ea]

(a) Let \( B \) be defined as the matrix

\[
B = \begin{bmatrix}
0 & 1 \\
1 & 0 & 1 \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots & 1 \\
& & & \ddots & 1 \\
& & & & 1 & 0
\end{bmatrix}.
\]

Using trigonometric identities, verify that the eigenvalues \( \lambda_i \) and eigenvectors \( v_i \) of \( B \) are

\[
\lambda_i = 2 \cos \left( \frac{i \pi}{N+1} \right), \quad v_i = \begin{bmatrix}
\sin \left( \frac{i \pi}{N+1} \right) \\
\sin \left( \frac{2i \pi}{N+1} \right) \\
\vdots \\
\sin \left( \frac{(N-1)i \pi}{N+1} \right) \\
\sin \left( \frac{Ni \pi}{N+1} \right)
\end{bmatrix}, \quad i = 1, \ldots, N.
\]

(Note: some of you may remember this problem from CAAM 335)

(b) Show that if \( v \) is an eigenvector of a matrix \( A \) with eigenvalue \( \lambda_A \) and an eigenvector of the matrix \( B \) with eigenvalue \( \lambda_B \), and \( \alpha \) is any real number then \( v \) is an eigenvector of the matrix \( G = A + \alpha B \) with eigenvalue \( \lambda_G = \lambda_A + \alpha \lambda_B \)

(c) The finite difference matrix \( A \) for the steady heat equation

\[
-\kappa \frac{\partial^2 u}{\partial x^2} = f(x)
\]

(with zero Dirichlet boundary conditions) can be written as

\[
A = \kappa \frac{1}{\Delta x^2} (2I d - B),
\]

where \( I d \) is the identity matrix. Determine the eigenvalues of \( A \) in terms of \( \kappa, h, \) and \( i, N \).
(d) Since $A$ is symmetric then we know from the spectral theorem (for matrices) that any vector $u \in \mathbb{R}^n$ can be expanded as a linear combination of eigenvectors of $A$ as

$$u = \sum_{j=1}^{N} \alpha_j v_j.$$ 

Use this representation to show that

$$A^nu = \sum_{j=1}^{N} \alpha_j \lambda_j^n v_j.$$ 

NOTE: the matrix $A^n$ is defined to be $AAA \ldots A$ where the matrix product is repeated $n$ times.

(e) If we discretize the time-dependent heat equation, with homogeneous Dirichlet boundary conditions, in space with central finite differences we arrive at the (semi-discrete) system of ordinary differential equations of the form:

$$\frac{\partial}{\partial t} U(t) = -AU(t)$$

with initial condition $U(0) = U_0$. Show that if we apply the Forward Euler method to this system of ODEs we ultimately arrive at the formula

$$U^{n+1} = (I - \Delta t A)^n U^0$$

(f) Using the results of part (d) and (e), explain why if $\Delta t > \frac{\Delta x^2}{2\kappa}$ will cause the solution to blow up (ie, tend to infinity) as $n \to \infty$.

NOTE: the condition on stability $\Delta t < \Delta x^2 / 2\kappa$ is called a CFL condition. This problem illustrates a method for deriving a CFL condition for the case when the matrix $A$ in part (e) is symmetric and a method like Forward or Backward Euler is used to discretize in time. This is the case when central finite differences are used to discretize the spatial derivative; however, this idea cannot be applied to a finite element discretization of the spatial derivative, even when the (spatial) mesh is uniform.

2. [30 pts, 5pts ea]

This problem has a significant amount of discussion before asking the questions; this is intended to help you explore the concepts, more thoroughly, over break. Note that throughout the discussion the value $\Delta t$ is assumed to be greater than zero. In the study of numerical approximations for ordinary differential equations the concept of stability is an important notion. A numerical approach to solving an ODE of the form

$$y'(t) = f(t, y(t))$$

$$y(0) = y_0$$

produces a sequence of iterates $y^k \approx y(t_k)$ where the approximation $y_k$ is a function of the, previously determined, vectors $y^{k-1}, y^{k-2}, \ldots, y^0, t$, and $\Delta t$. The concept of stability of a numerical method can be phrased, intuitively, as:

‘If we compute the sequence of numerical approximations $y^k$ corresponding to initial data $y_0$ and we compute the sequence of numerical approximations $\tilde{y}^k$ corresponding to initial data $\tilde{y}_0$ where $y_0$ and $\tilde{y}_0$ are close then the sequences of numerical approximations, $y^k$ and $\tilde{y}^k$, are also close’

We can make this rigorous with the following definition:
**Definition** A numerical method is *stable* if there exists $\delta > 0$, independent of $\Delta t$, such that $|y^k - \tilde{y}^k| \leq \delta |y_0 - \tilde{y}_0|$.

**Remark:** We briefly discussed Hadamard stability in class. One can easily show that Hadamard stability implies the above definition for linear numerical methods applied to linear ordinary differential equations. The numerical methods we discussed in class (Forward Euler, Backward Euler, and the Trapezoidal method) are all linear numerical methods.

There are other notions of stability that offer more nuance on this definition. One such definition is that of *A-stability*. Intuitively speaking, if we consider the differential equation

$$
y'(t) = ky(t)$$
$$y(0) = 1$$

where $k < 0$, we know that the exact solution is $y(t) = e^{kt}$, and $y(t) \to 0$ as $t \to \infty$; a numerical method is *A-stable* if the sequence of approximations it produces also exhibits this behavior. We can make this rigorous with the following definition:

**Definition** Consider the ordinary differential equation $y'(t) = ky(t)$ with $k < 0$. A numerical method is *A-stable* if it produces a sequence of approximations $y^1, y^2, \ldots, y^r$ such that $y^n \to 0$ as $n \to \infty$.

For certain classes of numerical methods it is not difficult to see if the method is A-stable. Suppose a numerical method produces a sequence of approximations $y^1, y^2, \ldots, y^r$ satisfying the relationship:

$$y^{n+1} = \phi(k \Delta t) y^n$$

Then applying this property repeatedly gives the relationship $y^{n+1} = \phi(k \Delta t)^n y^0$. We can see that such a method would be A-stable if and only if $|\phi(k \Delta t)| < 1$ for every $k < 0$, and every $\Delta t > 0$. Since we can analyze the A-stability of such a method by looking only at the function $\phi$ we give $\phi$ a special name.

**Definition:** Suppose a numerical method applied to the problem $y'(t) = ky(t)$, where $k < 0$, produces a sequence of approximations satisfying the relationship $y^{n+1} = \phi(k \Delta t) y^n$. Then we call $\phi(x)$ the *A-stability function* of the method.

If $\phi(x)$ is the A-stability function of some method then the set of all $x$ values satisfying $|\phi(x)| < 1$ is called the *stability set of $\phi$*. Note that if $x_0$ is an x-value such that $|\phi(x_0)| < 1$ then if we can choose $\Delta t$ such that $k \Delta t = x_0$ then our numerical method will produce approximations $y^n$ converging to zero as $n \to \infty$. This is due to the relationship $y^{n+1} = \phi(k \Delta t)^n y^0$ and the assumption that $k \Delta t = x_0$ is satisfied. Since $k < 0$ by assumption and the time step $\Delta t > 0$ we see that $k \Delta t = x_0$ is possible only in the case when $x_0 < 0$. Putting all of these notions together we can state the following result:

**Result:** A numerical method producing a sequence of approximations $y^1, y^2, \ldots, y^r$ where $y^{n+1} = \phi(k \Delta t) y^n$ will be stable if and only if the value $x_0 = k \Delta t$ is contained in the stability set of $\phi$. This implies that the method is A-stable if and if only the stability set of $\phi(x)$ contains the entire negative $x$ axis (since A-stability, by definition, means we should be stable for any values of $k < 0$ and $\Delta t > 0$ selected).

**Example:**
Suppose that we have a numerical method that we apply to the ODE $y'(t) = ky(t)$ resulting
in the formulat $y^{n+1} = 3k\Delta t y^n + y^n$. Re-arranging terms gives $y^{n+1} = (3k\Delta t + 1)y^n$ so that $\phi(k\Delta t) = 3k\Delta t + 1$. Therefore the stability function is found by making the substitution $k\Delta t = x$ which gives $\phi(x) = 3x + 1$. The stability set for this $\phi(x)$ is all of the $x$-values such that $|\phi(x)| < 1$. This means that $x$ satisfies

$$-1 < 3x + 1 < 1$$

So that $-2/3 < x < 0$. So some negative values of $x$ are in the stability set of $\phi$ but not every negative value of $x$. This means that our fictional numerical method would not be A-stable. Furthermore we see that the method IS stable only when $k\Delta t = x$ satisfies the inequality

$$-2/3 < k\Delta t < 0$$

Using the fact that $k < 0$ (a starting hypothesis) we see that this implies that $0 < |k|\Delta t < 2/3$ or, equivalently, $0 < \Delta t < 2/(3|k|)$.

The above example is a good motivation for the following definition:

**Definition** We say that a method is **conditionally stable** if stability relies on a relationship between $\Delta t$ and $k$. We say that a method is **unconditionally stable** if it is stable for any value of $k < 0$ and $\Delta t > 0$.

Thus we can see directly from the above definition that that an A-stable method is unconditionally stable.

**QUESTIONS**

Recall that the forward Euler method is:

$$y^{n+1} = y_n + dt f(t_n, y^n)$$

The backward Euler method is

$$y^{n+1} = y_n + dt f(t^{n+1}, y^{n+1})$$

and the trapezoidal method is

$$y^{n+1} = y_n + dt \frac{1}{2} \left( f(t^n, y^n) + f(t^{n+1}, y^{n+1}) \right)$$

If we apply any of these methods to the ODE

$$y'(t) = ky(t)$$

$$y(0) = y^0$$

where $k < 0$ we get a relationship of the form

$$y^{n+1} = \phi(k\Delta t)y^n$$

which implies that

$$y^{n+1} = \phi(k\Delta t)^n y^0$$

(a)  
   i. Show that the A-stability function of the forward Euler method is $\phi(x) = 1 + x$
   
   ii. Show that the A-stability function of the backward Euler method is $\phi(x) = 1/(1 - x)$

   iii. Show that the A-stability function of the trapezoidal method is $\phi(x) = (1 + (x/2))/(1 - (x/2))$
(b)  i. Show that the forward Euler method is conditionally stable and that $\Delta t < 2/|k|$ must be satisfied for stability (this proves that the forward Euler method is not A-stable).

   ii. Show that the backward Euler method is A-stable (this proves that backward Euler is unconditionally stable)

   iii. Show that the trapezoid method is A-stable (this proves the trapezoid method is unconditionally stable)