Intro. to time-dependent problems.

So far we have developed numerical methods for approximating solutions to the steady-state heat problem:

\[-\frac{d}{dx}(k(x) \frac{du}{dx}) = f(x)\]

by various boundary conditions.

**Methods**

1) **Finite difference** = approx. derivative via Taylor

2) **Spectral** \((k=1)\) = find an eigenbasis for the solution space.

3) **Finite element** = piecewise approximations.

Now we finally add time.

**Time-dependent heat problem**:

\[ \frac{du}{dt} - \frac{d}{dx}(k(x) \frac{du}{dx}) = f(x) \]

The solution \(u\) is a function of \(x, t\). \(u(x, t)\) is what we are looking to approximate.

Besides the PDE, what other information do we need? BOUNDARY conditions, \(u(0, t) = u(1, t) = 0\)

initial condition \(u(x, 0) = u_0(x)\)

**Ex:** Initial BVP

\[ \frac{du}{dt} - \frac{d^2u}{dx^2} = \sin(xt) \]

**BC:** \(u(0, t) = u(1, t) = 0\)

**IC:** \(u(x, 0) = 15 \sin(5\pi t x)\)

**Time:** 0
For both finite element & spectral, we approximate the Solution w/an expansion: \[ \hat{u}(x) = \sum_{j=1}^{N} a_j \varphi_j(x) \]

Applying this to a time dependent problem at time \( t \), we get:

\[ t = t_1 \Rightarrow \hat{u}(x, t_1) = \sum_{j=1}^{N} (a_j)_{t_1} \varphi_j(x) \]

\[ t = t_2 \Rightarrow \hat{u}(x, t_2) = \sum_{j=1}^{N} (a_j)_{t_2} \varphi_j(x) \]

For smoothly varying in time, for each \( j \), the coefficient is a function of time \( a_j(t) \).

Thus we will seek a solution of the form:

\[ \hat{u}(x, t) = \sum_{j=1}^{N} a_j(t) \varphi_j(x) \]

Apply the spectral or finite element method to DE:

\[ \frac{du}{dt} = \frac{d^2 u}{dx^2} = f \]

And find \( \hat{a}(t) \) w/ entries \( a_j(t) \). Is the solution \( \hat{a} \)

\[ \frac{d\hat{a}}{dt} = A\hat{a} + f(t) \]

where \( A \) is some matrix depending on method used.

This is a linear, ODE system.

Using finite differences to approximate the space derivative results in something similar

\[ \frac{du(x, t)}{dt} = \frac{u(x_{i+1}, t) - 2u(x_i, t) + u(x_{i-1}, t)}{h^2} = f(x_i, t) \]

Time \( t \)
\[ \frac{d\vec{u}}{dt} = A\vec{u} + \vec{f}(t) \quad \text{where} \quad \vec{u}(t) \quad \text{and} \quad \vec{f}(t) \]

[\vec{u}(t)]_j = u(x_j, t)

Again this is a linear ODE system.

Homogeneous Linear System of ODES

We will first consider the case \( f(t) = 0 \). Then the ODE problem is to find \( \vec{u}(t) \) so that

\[ \frac{d\vec{u}}{dt} = A\vec{u}(t) \quad \text{where} \quad A \text{ is some fixed matrix.} \]