Model time-dependent IBVP: Fourier Series (spectral method)

Consider the model IBVP given by:

\[ pc \frac{du}{dt} - k \frac{\partial^2 u}{\partial x^2} = f(x,t) \quad 0 < x < l \quad t > 0 \quad \text{(DE)} \]

\[ u(x,0) = g(x) \quad 0 < x < l \quad \text{(Initial condition)} \]

\[ u(0,t) = 0 \quad t > 0 \]

\[ u(l,t) = 0 \quad t > 0 \]

To start let's take \( f(x,t) = 0 \). So \( k = 1 \).

Assume the solution \( u(x,t) \) is separable, i.e.,

\[ u(x,t) = V(t) W(x) \]

where \( V(t) \) is a function just of \( t \),

\( W(x) \) is a function just of \( x \).

Plug this into the DE.

\[ pc V'(t) W(x) - V(t) W''(x) = 0. \]

Now let's collect terms:

\[ \frac{pc V'(t)}{V(t)} = \frac{W''(x)}{W(x)} \quad \text{function of } t \]

\[ \quad \text{function of } x. \]

Only way these can be equal is if the ratios equal a constant.
i.e. \( \frac{\partial c}{\partial t} = \frac{W''(x)}{W(x)} = \lambda \), for \( t \to \infty \)

\( \Rightarrow \) We need to solve two ODES.

1. \( W''(x) = \lambda \cdot W(x) \)
2. \( \frac{\partial c}{\partial t} = \lambda \cdot V(t) \)

1 looks familiar. 2 does not look hard to solve.

Let's transform IC & BC's to conditions on \( V \) & \( W \).

IC. \( U(x, t_0) = W(x) \cdot V(t_0) = \psi(x) \). Not very helpful.

but BC. \( U(0, t) = W(0) \cdot V(t) = 0 \)

\( \Rightarrow W(0) = 0 \).

\( U(l, t) = W(l) \cdot V(t) = 0 \)

\( \Rightarrow W(l) = 0 \).

\( \Rightarrow \) The Problem to solve for \( W \) is

\[
\begin{cases}
    -W''(x) + \lambda W(x) = 0 \\
    W(0) = W(l) = 0.
\end{cases}
\]

We know from the spectral method that the eigenfunctions from this problem are \( w_n(x) = \sin \left( \frac{n\pi x}{l} \right) \) with eigenvalue \( \lambda_n = \frac{n^2 \pi^2}{l^2} \).

\( \Rightarrow U(x, t) = \sum_{n=1}^{\infty} V_n(t) \sin \left( \frac{n\pi x}{l} \right) \)
where $V_n(t)$ is the solution of

$$V_n'(t) = \lambda_n V_n(t) \quad \text{for } t > t_0.$$ 

Take another look at IC.

$$u(x,t_0) = \sum_{n=1}^{\infty} V_n(t_0) \sin \left(\frac{n\pi x}{e}\right) = \psi(x).$$

$\Rightarrow V_n(t_0)$ are the Fourier coefficients of $\psi(x)$.

i.e. $V_n(t_0) = \frac{2}{e} \int_0^e \psi(x) \sin \left(\frac{n\pi x}{e}\right) dx = \psi_n$

So for each $n$ we need to solve the IVP

\[
\begin{cases}
V_n'(t) = \lambda_n V_n(t) \\
V_n(t_0) = \psi_n
\end{cases}
\]

* is a separable ODE

$$\int \frac{V_n'(s)}{V_n(s)} ds = \lambda_n \int_{t_0}^{t} 1 ds.$$ 

$\Rightarrow \ln(V_n(t)) - \ln(V_n(t_0)) = \lambda_n (t-t_0)$

$\Rightarrow V_n(t) = Ce^{\lambda_n(t-t_0)}$

Use IC $V_n(t_0) = \psi_n = \psi_n$.

$\Rightarrow V_n(t) = \psi_n e^{\lambda_n(t-t_0)}$

$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \psi_n e^{\lambda_n(t-t_0)} \sin \left(\frac{n\pi x}{e}\right)$

where

$$\lambda_n = \left(\frac{n\pi}{e}\right)^2, \quad \psi_n = \frac{2}{e} \int_0^e \psi(x) \sin \left(\frac{n\pi x}{e}\right) dx.$$
What happens if \( f(x,t) \neq 0 \)?

We use the fixed case as inspiration.

Assume our solution has a Fourier series representation in space, i.e.

\[ u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin \left( \frac{n\pi x}{l} \right) \]

Also assume we can write \( f(x,t) \) in the same expansion form.

\[ f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin \left( \frac{n\pi x}{l} \right) \]

Plug this into the PDE to find ODE for \( \frac{\partial^2}{\partial t^2} \left( \sum_{n=1}^{\infty} a_n(t) \sin \left( \frac{n\pi x}{l} \right) \right) \).

\[ \rho C \frac{\partial}{\partial t} \left( \sum_{n=1}^{\infty} a_n(t) \sin \left( \frac{n\pi x}{l} \right) \right) - k \frac{\partial^2}{\partial x^2} \left( \sum_{n=1}^{\infty} a_n(t) \sin \left( \frac{n\pi x}{l} \right) \right) = \sum_{n=1}^{\infty} f_n(t) \sin \left( \frac{n\pi x}{l} \right) \]

\[ \Rightarrow \rho C \sum_{n=1}^{\infty} a_n'(t) \sin \left( \frac{n\pi x}{l} \right) - k \sum_{n=1}^{\infty} \left( \frac{n\pi}{l} \right)^2 a_n(t) \sin \left( \frac{n\pi x}{l} \right) = \sum_{n=1}^{\infty} f_n(t) \sin \left( \frac{n\pi x}{l} \right) \]

\[ \Rightarrow \sum_{n=1}^{\infty} \left[ \rho C a_n'(t) - k \left( \frac{n\pi}{l} \right)^2 a_n(t) - f_n(t) \right] \sin \left( \frac{n\pi x}{l} \right) = 0. \]

Take inner product with \( \sin \left( \frac{m\pi x}{l} \right) \) to find

\[ \rho C \cdot a_m'(t) - k \lambda_m a_m(t) = f_m(t). \]
Now we need the initial condition. Like before we turn to the Fourier representation of the initial condition

$$u(x, t_0) = \sum_{n=1}^{\infty} \psi_n \sin \left( \frac{n\pi x}{l} \right) = \sum_{n=1}^{\infty} a_n(t_0) \sin \left( \frac{n\pi x}{l} \right)$$

Take inner product of the series equation w/ \( \sin \left( \frac{m\pi x}{l} \right) \) to find

$$a_m(t_0) = \psi_m$$

\( \Rightarrow \) to find \( a_m(t) \) we needed to solve the following ODE

$$\begin{cases} pc a_m'(t) - k_m a_m(t) = f_m(t) \\ a_m(t_0) = \psi_m \end{cases}$$

The solution to this ODE is

$$a_m(t) = \psi_m e^{\frac{pc}{k_m}(t-t_0)} + \frac{1}{pc} \int_{t_0}^{t} e^{\frac{pc}{k_m}(t-s)} f_m(s) ds$$
0 Steps for the Spectral in time method

Consider the IBVP

\[ \begin{cases} \alpha \frac{du}{dt} + Lu = f(t) & 0 < x < L \\ u(x,0) = g(x) \\ \text{Boundary Conditions} \end{cases} \]

where \( L = \frac{\partial^2}{\partial x^2} \)

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Step 1: Find the eigenfunctions \( \phi_n \) and eigenvalues \( \lambda_n \) for the steady state problem

\[ \begin{cases} Lu = \lambda u \\ BC \end{cases} \]

Step 2: Express the solution \( u \) of the IBVP \( f \) as a Fourier series in space with this basis

\( u(x,t) = \sum_{n=1}^{\infty} a_n(t) \phi_n(x) \)

\( f(x,t) = \sum_{n=1}^{\infty} f_n(t) \phi_n(x) \)

Step 3: Find the coefficients of \( f \).

\[ f_n(t) = \int_0^L f(x,t) \phi_n(x) \, dx \]

Step 4: WRITE Initial condition in Fourier series form.

\[ g(x) = \sum_{n=1}^{\infty} g_n \phi_n(x) \]

where \( g_n = \int_0^L g(x) \phi_n(x) \, dx \).
Step 5: Plug series into DE. to get

\[ \sum_{n=1}^{\infty} c_n \frac{d\phi_n(x)}{dt} + \lambda_n \phi_n(x) = \sum_{n=1}^{\infty} f_n(t) \phi_n(x) \]

Step 6: Take inner product of both sides wrt \( \phi_m(x) \) to get an ODE for \( a_m(t) \).

\[ p c \frac{da_m(t)}{dt} + \lambda_m a_m(t) = f_m(t). \]

Step 7: Use initial condition on \( u \) to get initial condition for \( a_m \)

\[ u(x,t_0) = \sum_{m=1}^{\infty} a_m(t_0) \phi_n(x) = \sum_{n=1}^{\infty} g_m \phi_n(x) \]

take inner product w/ \( \phi_m \) to get.

\[ a_m(t_0) = g_m \]

Step 8: Consider the IVP.

\[ \begin{cases} p c \frac{d^2 a_m(t)}{dt^2} + \lambda_m a_m(t) = f_m(t) \\ a_m(t_0) = g_m \end{cases} \]

This has solution

\[ a_m(t) = g_m e^{\frac{-\lambda_m(t-t_0)}{pc}} + \frac{1}{pc} \int_{t_0}^{t} e^{\frac{-\lambda_m(t-s)}{pc}} f_m(s) \, ds \]

Step 9: Write solution

\[ u(x,t) = \sum_{n=1}^{\infty} a_n(t) \phi_n(x). \]
Example: consider the problem.

\[ \frac{pc}{c} \frac{du}{dt} + k \frac{d^2 u}{dx^2} = 0 \quad 0 < x < 50 \]

\[ u(x, t_0) = 5 - \frac{1}{5} (x - 25) \]

\[ u(0, t) = u(l, t) = 0 \]

Plot of \( u(x, t) \)