CAAM 353: Computational Numerical Analysis

Pledged Homework 4, Jan. 31, 2008

Due: Feb. 7 , 2008

This homework is pledged! You may use your notes and books for this homework. However, you are not allowed to discuss this homework assignment with another person, except your instructor. Write out and sign the pledge.

Note: All MATLAB *functions mentioned in this homework assignment can be found on the CAAM353 homepage, or come with* MATLAB . *Turn in all* MATLAB *code that you have written and turn in all output generated by your* MATLAB *functions/scripts.* MATLAB *functions/scripts must be commented and output must be formatted nicely.*

Problem 1 (10 points)

- The textbook formula for the two roots of the quadratic equation $ax^2 + bx + c = 0$ is given by
$$\left(-b \pm \sqrt{b^2 - 4ac}\right)/(2a). \tag{1}$$
Prove that
$$x_1 = \left(-b - \text{sign}(b)\sqrt{b^2 - 4ac}\right)/(2a), \quad x_2 = c/(ax_1) \tag{2}$$
is a mathematically equivalent representation of the two roots.
- Use both formulas (1) and (2) to compute the two roots of the quadratic equation with
$$a = 5 * 10^{-6}, b = 100, c = 5 * 10^{-7}$$
and with
$$a = 5 * 10^{-6}, b = -100, c = 5 * 10^{-7}$$
Which formula do you prefer and why?

Problem 2 (10 points) Show how to rewrite the following expresions to avoid catastrophic cancellation for the indicated arguments.

i. $\sqrt{x+1} - 1$, $x \approx 0$,

ii. $(1 - \cos(x))/\sin(x)$, $x \approx 0$.

- Demonstrate that catastrophic cancellation occurs when the formulas above are used and
- Compute the absolute and relative errors between the results computed with the formulas above and your formula.

Problem 3 (35 points) Let $g : I\!R \to I\!R$ be three times continuously differentiable. The second derivative of g at x can be approximated using

$$g''(x) \approx \frac{g(x+\delta) - 2g(x) + g(x-\delta)}{\delta^2}. \tag{3}$$

i. (10points) Taylor expansion of g around x gives the identities

$$\begin{aligned} g(x+\delta) &= g(x) + g'(x)\delta + \frac{1}{2}g''(x)\delta^2 + \frac{1}{6}g'''(x+\theta_+\delta)\delta^3, \\ g(x-\delta) &= g(x) - g'(x)\delta + \frac{1}{2}g''(x)\delta^2 - \frac{1}{6}g'''(x-\theta_-\delta)\delta^3 \end{aligned}$$

for some $\theta_+, \theta_- \in [0,1]$. Assume that

$$|g'''(x+\theta\delta)| \le M \quad \text{for all } \theta \in [-1,1].$$

Use this Taylor expansion to derive an upper bound for the error

$$\left| g''(x) - \frac{g(x+\delta) - 2g(x) + g(x-\delta)}{\delta^2} \right|.$$

ii. (5points) Suppose that instead of $g(x-\delta)$, $g(x)$, $g(x+\delta)$ we only have approximate function values $g_\varepsilon(x-\delta)$, $g_\varepsilon(x)$, $g_\varepsilon(x+\delta)$ available, where

$$|g(x+\delta) - g_\varepsilon(x+\delta)| \le \varepsilon, \quad |g(x) - g_\varepsilon(x)| \le \varepsilon, \quad |g(x-\delta) - g_\varepsilon(x-\delta)| \le \varepsilon.$$

Thus, instead of (3) we use

$$g''(x) \approx \frac{g_\varepsilon(x+\delta) - 2g_\varepsilon(x) + g_\varepsilon(x-\delta)}{\delta^2}.$$

Derive an upper bound for the error

$$\left| g''(x) - \frac{g_\varepsilon(x+\delta) - 2g_\varepsilon(x) + g_\varepsilon(x-\delta)}{\delta^2} \right|. \tag{4}$$

iii. (10points) Compute the step-size δ_* that minimizes the upper bound for (4) you have derived in ii.

Compute the value of the upper bound for (4) at δ_*.

iv. (5points) Let $g(x) = \exp(x)$. Use (3) with $\delta = 10^{-i}$, $i = 1, \ldots, 20$, to compute approximations of $g''(x)$ at $x = 1$.

Plot the error between the second derivative and its approximation on a log-log-scale.

v. (5points) Carefully explain the results you have obtained in iv.

Problem 2 (10 points) The power series for $\sin(x)$ is

$$\sin(x)x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

This Matlab function uses the series to compute $\sin(x)$

```
function s = powersin(x)
% POWERSIN. Power series for sin(x).
% POWERSIN(x) tries to compute sin(x)
% from a power series
s = 0;
t = x;
n = 1;
while s+t ~= s;
     s = s + t;
     t = -x.^2/((n+1)*(n+2)).*t;
     n = n + 2;
end
```

- What causes the while loop to terminate?
- Answer the following questions for $x = \pi/2, 11\pi/2, 21\pi/2, 31\pi/2$:
  - How accurate is the computed result?
  - How many terms are required?
  - What is the largest term in the series?