

CAAM 353: Computational Numerical Analysis

Homework 5, Feb. 7, 2008

Due: Feb. 14, 2008

Note: All MATLAB functions mentioned in this homework assignment can be found on the CAAM353 homepage, or come with MATLAB. Turn in all MATLAB code that you have written and turn in all output generated by your MATLAB functions/scripts. MATLAB functions/scripts must be commented and output must be formatted nicely.

Problem 1 (22 points) Recall that the relative error between the solutions of the linear systems $Ax = b$ and $A\hat{x} = \hat{b}$ obeys

$$\frac{\|\hat{x} - x\|_p}{\|x\|_p} \leq \kappa_p(A) \frac{\|\hat{b} - b\|_p}{\|b\|_p}. \quad (1)$$

i. (6points) Let

$$A = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Compute the matrix norms $\|A\|_p$ and the condition numbers $\kappa_p(A) = \|A\|_p \|A^{-1}\|_p$ for $p = 1, 2, \infty$.

ii. (4points) Let $b = (5, 1.02, 1.04, 1.1)^T$. Compute \hat{b} by rounding the entries of b to the nearest integers. Compute the solution \hat{x} of $A\hat{x} = \hat{b}$.

iii. (6points) Use (1) to compute upper bounds for the relative errors $\|\hat{x} - x\|_p / \|x\|_p$, $p = 1, 2, \infty$. Do not compute the solution x of $Ax = b$.

iv. (6points) Compute the solution x of $Ax = b$ and the relative errors $\|\hat{x} - x\|_p / \|x\|_p$, $p = 1, 2, \infty$.

Do your computations agree with the error bound (1).

Problem 2 (13 points)

Let

$$A = \begin{pmatrix} 1.000 & 1.001 \\ 1.000 & 1.000 \end{pmatrix}.$$

i. (5points) The exact solution of $Ax = b_1$ with $b_1 = (2.001, 2.000)^T$ is $x_1 = (1, 1)^T$.

Compute the solution \hat{x}_1 of $Ax = \hat{b}_1$, where

$$\hat{b}_1 = \begin{pmatrix} 2.002 \\ 2.000 \end{pmatrix}.$$

Compute $\|x_1 - \hat{x}_1\|_\infty$ and $\|b_1 - \hat{b}_1\|_\infty$.

ii. (5points) The exact solution of $Ax = b_2$ with $b_2 = (1, 0)^T$ is $x_2 = (-1000, 1000)^T$.

Compute the solution \hat{x}_2 of $Ax = \hat{b}_2$, where

$$\hat{b}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Compute $\|x_2 - \hat{x}_2\|_\infty$ and $\|b_2 - \hat{b}_2\|_\infty$.

iii. (3points) Note that the residual $\|b_1 - \hat{b}_1\|_\infty$ is small relative to $\|b_1\|_\infty$ while $\|x_1 - \hat{x}_1\|_\infty$ is large relative to $\|x_1\|_\infty$. On the other hand, $\|b_2 - \hat{b}_2\|_\infty$ is large relative to $\|b_2\|_\infty$ while $\|x_2 - \hat{x}_2\|_\infty$ is small relative to $\|x_2\|_\infty$.

Do your results agree with (1)? (Compare left and right hand side of (1) for both cases.)

Problem 3 (20 points) This problem illustrates the need or partial pivoting in the LU-decomposition,

Consider the linear system $Ax = b$ with

$$A = \begin{pmatrix} 20 & -7 & 0 \\ -6 & 2.099 & 12 \\ 10 & -1 & 10 \end{pmatrix}, \quad b = \begin{pmatrix} 7 \\ 3.901 \\ 6 \end{pmatrix}$$

The exact solution of the linear system is

$$x = (0, -1, 1/2).$$

- i. (10points) • Disable pivoting in `lu_pp_fl.m`, i.e., enforce $ipiv(k) = k, k = 1, \dots, n-1$, and call the resulting function `lu_np_fl.m`. Use `lu_np_fl.m` and `lu_pp_sl_fl.m` to solve the linear system above using the LU decomposition *without* pivoting in a floating point number system with base $\beta = 10$ and mantissa length $m = 5$. What is the error between the computed and the exact solution? (The error should be computed in double precision arithmetic, the MATLAB default.)
- Use `lu_pp_fl.m` and `lu_pp_sl_fl.m` to solve the linear system above using the LU decomposition *with* pivoting, again in a floating point number system with base $\beta = 10$ and mantissa length $m = 5$. What is the error between the computed and the exact solution in this case?

In both cases print the arrays A and $ipiv$ returned by `lu_np_fl.m` or `lu_pp_fl.m`, respectively, print the computed solution and the error.

ii. (10 points) Use `getLUP.m` to generate L, U, P from the outputs of `lu_np_fl.m` and `lu_pp_fl.m`, respectively. Print L, U, P . Note that the entries ℓ_{ij} of matrix L computed with `lu_pp_fl.m` obey $|\ell_{ij}| \leq 1$.

In both cases compute (in double precision arithmetic) $P^T LU$ and $\|A - P^T LU\|_2$.

In both cases compute (in double precision arithmetic) $b - Ax_{\text{computed}}$ and $\|b - Ax_{\text{computed}}\|_2$.

What is, in this case, the cause of the large error between the computed solution using `lu_np_fl.m` and `lu_pp_sl_fl.m` and the exact solution?