CAAM 353: Computational Numerical Analysis

Homework 8, March 13, 2008

Due: March 20, 2008

Note: All MATLAB *functions mentioned in this homework assignment can be found on the CAAM353 homepage, or come with* MATLAB . *You can use the* MATLAB *codes posted on the CAAM353 web-page. If you modify these codes, please turn in the modified code. Otherwise you do not have to turn in printouts of the codes posted on the CAAM353 web-page. Turn in all* MATLAB *code that you have written and turn in all output generated by your* MATLAB *functions/scripts.* MATLAB *functions/scripts must be commented, output must be formatted nicely, and plots must be labeled.*

Problem 1 (25 points) In laying water mains, utilities must be concerned with the possibility of freezing. Although solid and weather conditions are complicated, reasonable approximations can be made on the basis of the assumption that soil is uniform in all directions. In that case the temperature in degrees Celsius $T(x,t)$ at a distance x (in meters) below the surface, t seconds after the beginning of a cold snap, approximately satisfies

$$\frac{T(x,t) - T_s}{T_i - T_s} = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right),$$

where T_s is the constant temperature during a cold period, T_i is the initial soil temperature before the cold snap, α is the thermal conductivity (in meters^2 per second), and

$$\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t \exp(-s^2) ds$$

Assume that $T_i = 20$ [degress C], $T_s = -15$[degress C], $\alpha = 0.138 \cdot 10^{-6}$ [meters^2 per second].

i. (10points) We want to determine how deep a water main should be buried so that it will only freeze after 60 *days* expose at this constant surface temperature.

   Formulate the problem as a root finding problem $f(x) = 0$. What is f and what is f'?

   Plot the function f on $[0, \bar{x}]$, where $\bar{x}$ is chosen so that $f(\bar{x}) > 0$.

ii. (5points) Compute an approximate depth using the Bisection Method with starting values $a_0 = 0$[meters] and $b_0 = \bar{x}$ [meters].

iii. (10points) Compute an approximate depth using Newton's Method with starting value $x_0 = 0.01$[meters].

   What happens if you start with $x_0 = \bar{x}$?

Problem 2 (20 points)

A long conducting rod of diameter D meters and electrical resistance R per unit length is in a large enclosure whose walls (far away from the rod) are kept a temperature T_s degrees C. Air flows past the rod at temperature T_∞ degrees C. If an electrical current I passes through the rod, the temperature of the rod eventually stabilizes to T, where T satisfies

$$f(T) = \pi D h(T - T_\infty) + \pi D \varepsilon \sigma (T^4 - T_s^4) - I^2 R = 0, \tag{1}$$

where

$$\begin{aligned}
\sigma &= \text{Stefan–Boltzman constant} = 5.67 \cdot 10^{-8} \text{ Watts/meter}^2\text{Kelvin}^4, \\
\varepsilon &= \text{rod surface emissivity} = 0.8, \\
h &= \text{heat transfer coefficient of air flow} = 20 \text{ Watts/meter}^2\text{Kelvin}, \\
T_\infty &= T_s = 25\ {}^0\text{C}, \\
D &= 0.1 \text{ meter}, \\
I^2R &= 100.
\end{aligned}$$

Be aware of the different units! (K = ^{0}C+273.15)

Equation (1) is a root finding problem for $f(T) = \pi D h(T - T_\infty) + \pi D \varepsilon \sigma (T^4 - T_s^4) - I^2 R$.

i. (5points) Plot f for T between 200K and 400K.

ii. (5points) Compute an approximate steady state temperature using the Bisection Method with starting values $a_0 = 200$K and $b_0 = 400$K.

iii. (10points) Compute an approximate steady state temperature using Newton's Method with starting value 200K.

Problem 3 (30 points)

If an amount a is borrowed at interest rate r for n years, then the total amount to be repaid is given by

$$a(1+r)^n.$$

Yearly payments of p each would reduce this amount by

$$\sum_{i=0}^{n-1} p(1+r)^i = p\frac{(1+r)^n - 1}{r}.$$

The loan will be repaid when these two quantities are equal.

i. (10points) For a loan of $a = \$100,000$ and yearly payments of $p = \$10,000$, how long will it take to pay off the loan if the interest rate is 6 percent, i.e., $r = 0.06$?

ii. (10points) For a loan of $a = \$100,000$ and yearly payments of $p = \$10,000$, what interest rate r would be required for the loan to be paid off in $n = 20$ years?

iii. (10points) For a loan of $a = \$100,000$, how large must the yearly payments p be for the loan to be paid off in $n = 20$ years at 6 percent interest?

You may use any method you like to solve the given equation in each case. For the purpose of this problem, we will treat n as a continuous variable, i.e., it can have non-integer values.