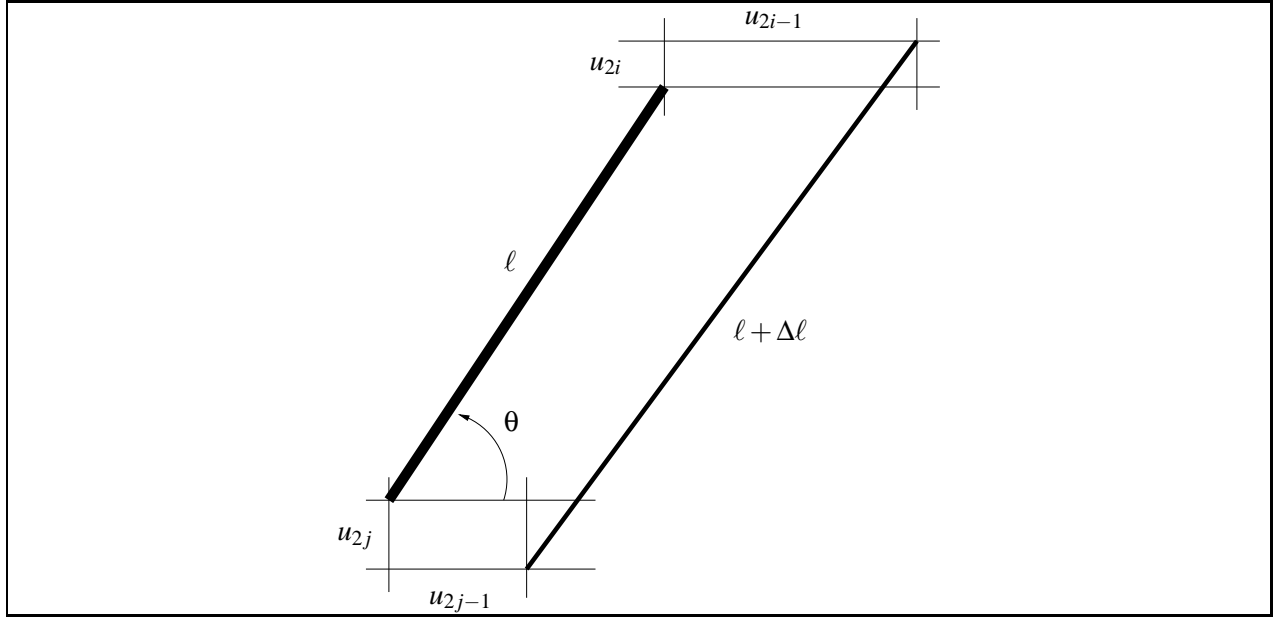


## 2.9 Planar Trusses

Our derivation of the equations for a planar truss follows [Str86, Sec. 2.4], but we use a different notation.

We consider trusses with  $m$  bars and  $n$  nodes. Each node can be displaced in horizontal and vertical direction. If the node number is  $j$ , then its horizontal displacement is denoted by  $u_{2j-1}$  and its vertical displacement is denoted by  $u_{2j}$ . We use the following convention: The horizontal displacement  $u_{2j-1}$  is positive if node  $j$  is moved right; the vertical displacement  $u_{2j}$  is positive if node  $j$  is moved down.



**Figure 2.5:** Elongation of a generic bar.

Now consider the generic bar in Figure 2.5. If  $\ell$  is the length of the undeformed bar and  $\ell + \Delta\ell$  is the length of the deformed bar, then, by Pythagoras' theorem,

$$\begin{aligned}
 (\ell + \Delta\ell)^2 &= (\ell \cos(\theta) + u_{2i-1} - u_{2j-1})^2 + (\ell \sin(\theta) + u_{2i} - u_{2j})^2 \\
 &= \ell^2 + 2\ell(\cos(\theta)u_{2i-1} - \cos(\theta)u_{2j-1} + \sin(\theta)u_{2i} - \sin(\theta)u_{2j}) \\
 &\quad + (u_{2i-1} - u_{2j-1})^2 + (u_{2i} - u_{2j})^2 \\
 &= \ell^2 \left( 1 + 2(\cos(\theta)u_{2i-1} - \cos(\theta)u_{2j-1} + \sin(\theta)u_{2i} - \sin(\theta)u_{2j})/\ell \right. \\
 &\quad \left. + [(u_{2i-1} - u_{2j-1})^2 + (u_{2i} - u_{2j})^2]/\ell^2 \right).
 \end{aligned}$$

Now we take square roots, use the Taylor expansion  $\sqrt{1+t} = 1 + t/2 + O(t^2)$ , and neglect the terms that are quadratic in the  $u_k$ 's to obtain

$$\ell + \Delta\ell \approx \ell + \cos(\theta)u_{2i-1} - \cos(\theta)u_{2j-1} + \sin(\theta)u_{2i} - \sin(\theta)u_{2j}.$$

If the displacements are small relative to the length of the undeformed bar, then the previous approximation of  $\ell + \Delta\ell$  is very good and we will use it as an equality to derive the equation

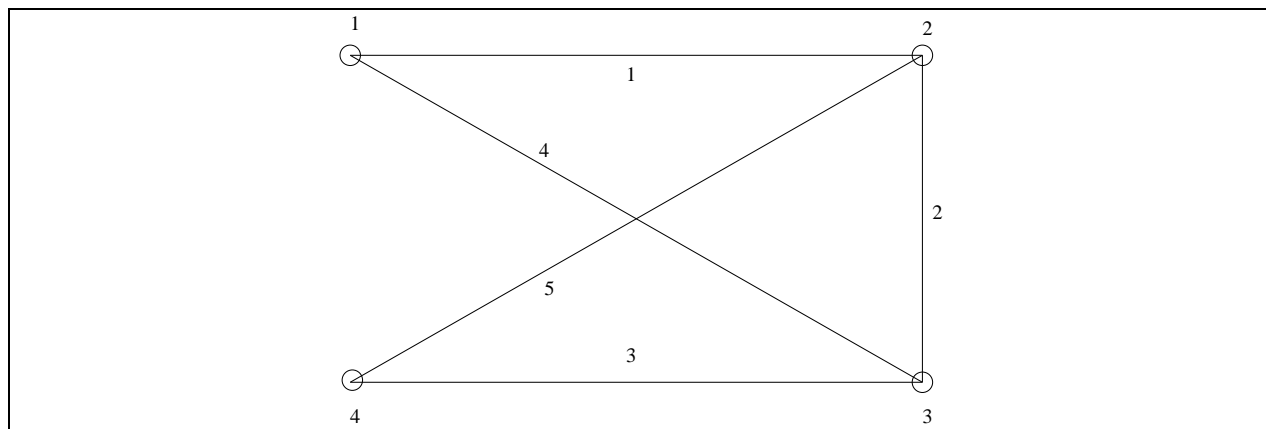
$$\Delta\ell = \cos(\theta)u_{2i-1} - \cos(\theta)u_{2j-1} + \sin(\theta)u_{2i} - \sin(\theta)u_{2j} \quad (2.37)$$

for the elongation  $\Delta\ell$  of the bar. The bar is stretched if  $\Delta\ell > 0$  and it is compressed if  $\Delta\ell < 0$ . This is compatible with our convention for the sign of the displacements  $u_{2j}$  and  $u_{2j-1}$  if we take the angle  $\theta$  with a negative sign.

The relation (2.37) between elongation and displacement is used for all bars. This gives the linear system

$$B^T u = \Delta\ell. \quad (2.38)$$

For example, consider the truss in Figure 2.6. To apply (2.37) for each bar, we have to decide at which end



**Figure 2.6:** Truss with  $N = 4$  nodes and  $m = 5$  bars.

of the bar we take the angle, i.e., which end of the bar corresponds to node  $j$  and to node  $i$  in the generic bar in Figure 2.5. We always take the angle at the lower left node of the bar (at the left node if the bar is horizontal). If we apply relation (2.37) for all bars and follow our convention, we obtain the following matrix  $B^T$ :

$$\begin{pmatrix} -\cos(0) & -\sin(0) & \cos(0) & \sin(0) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\frac{-\pi}{2}) & \sin(\frac{-\pi}{2}) & -\cos(\frac{-\pi}{2}) & -\sin(\frac{-\pi}{2}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(0) & \sin(0) & -\cos(0) & -\sin(0) \\ \cos(\frac{-3\pi}{4}) & \sin(\frac{-3\pi}{4}) & 0 & 0 & -\cos(\frac{-3\pi}{4}) & -\sin(\frac{-3\pi}{4}) & 0 & 0 \\ 0 & 0 & \cos(\frac{-\pi}{4}) & \sin(\frac{-\pi}{4}) & 0 & 0 & -\cos(\frac{-\pi}{4}) & -\sin(\frac{-\pi}{4}) \end{pmatrix}.$$

Thus equation (2.38) for the truss in Figure 2.6 is given by

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix} = \begin{pmatrix} \Delta\ell_1 \\ \Delta\ell_2 \\ \Delta\ell_3 \\ \Delta\ell_4 \\ \Delta\ell_5 \end{pmatrix}. \quad (2.39)$$

The displacement of the bars  $i = 1, \dots, m$  generate an internal forces  $q_i, i = 1, \dots, m$ , acting on the nodes. The forces can be composed into forces in horizontal and vertical direction. Our indexing of the forces acting

on the nodes follows the indexing of the nodes, i.e., the horizontal component of the force acting on node  $j$  is  $f_{2j-1}$  and the vertical component is  $f_{2j}$ . Moreover, a force applied in the right direction means that  $f_{2j-1} > 0$  and a force applied downward means that  $f_{2j} > 0$ . If the truss is in equilibrium, then the internal forces acting on a node must equal the external forces acting on the same node and this force balance must be true for all nodes. One can show, see [Str86, Sec. 2.4], that these force balances can be written as a system of linear equations given by

$$Bq = f. \quad (2.40)$$

Finally, we need a constitutive law that relates elongations to the internal forces. Let  $a_i$  be the cross sectional area of the undeformed bar. We define the stress  $\sigma_i$  and the strain  $\varepsilon_i$  as

$$\sigma_i = \frac{q_i}{a_i}, \quad \varepsilon_i = \frac{\Delta \ell_i}{\ell_i}.$$

The assumption of linear elasticity states that

$$\sigma_i = E_i \varepsilon_i,$$

where  $E_i$  is Young's modulus for the material of the  $i$ -th bar. If we set

$$D = \text{diag}(\dots, a_i E_i / \ell_i, \dots), \quad (2.41)$$

then

$$D\Delta \ell = q. \quad (2.42)$$

Let us discuss the units for the quantities we have introduced so far. The elongations  $\Delta \ell_i$  of the bars and the horizontal and vertical displacements  $u_{2j-1}$ ,  $u_{2j}$  of the nodes are given in units of length, e.g., [m]. The strains  $\varepsilon_i$  are dimensionless quantities and the stresses  $\sigma_i$  are given in units of force per area, e.g., Newton per square meter [ $\text{N}/\text{m}^2$ ], which is called a Pascal [Pa]. Hence, the Young's modulus  $E_i$  must also be given in units of force per area.

If we combine equations (2.38), (2.40), and (2.42) we obtain the equation

$$BDB^T u = f. \quad (2.43)$$

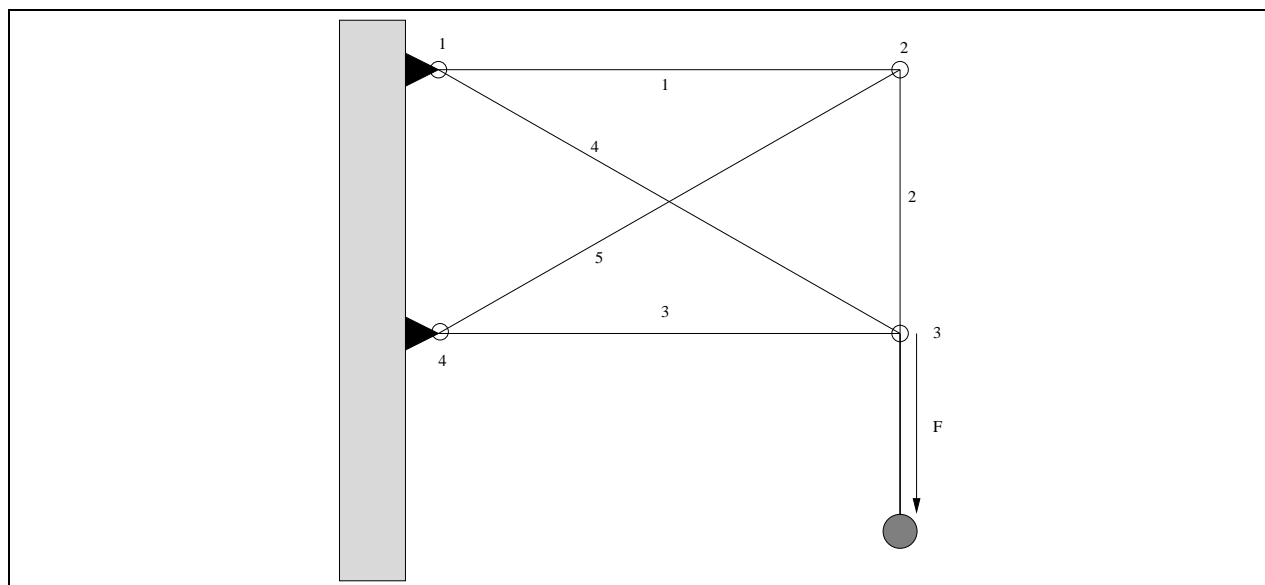
The matrix

$$K = BDB^T \quad (2.44)$$

is called the stiffness matrix of the structure.

If nodes in a truss are supported, then the corresponding displacement is deleted from the system. For example, consider the truss in Figure 2.7. The nodes 1 and 4 are supported. Equation (2.38) for this truss is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} = \begin{pmatrix} \Delta \ell_1 \\ \Delta \ell_2 \\ \Delta \ell_3 \\ \Delta \ell_4 \\ \Delta \ell_5 \end{pmatrix} \quad (2.45)$$



**Figure 2.7:** Truss with two supported and two unsupported nodes and five bars.

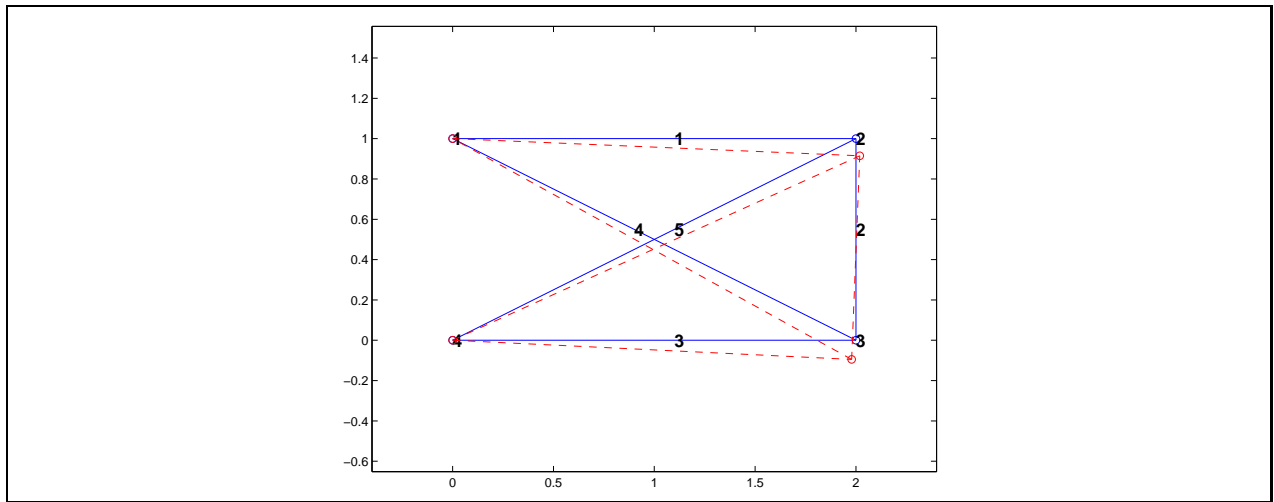
and the equilibrium equation (2.40) is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & -1 & 0 & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 0 & 1 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 1 & 0 & \frac{\sqrt{2}}{2} & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -F \end{pmatrix}. \quad (2.46)$$

The solution  $u$ , rounded to four digits, of the system (2.43) with  $B$  and  $f$  given in (2.46) and with  $D = \text{diag}(100, 100, 100, 100, 100)$  and  $F = 2$  is given by

$$\begin{pmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} = 10^{-2} \begin{pmatrix} 1.895 \\ 8.526 \\ -2.105 \\ 9.474 \end{pmatrix}.$$

These displacements lead to the deformed truss depicted in Figure 2.8. To obtain a more realistic picture, we assume that all bars in the truss are made of a stainless steel with Young's modulus  $E_i = 195$  GPa (giga Pascal =  $10^9$  N/m<sup>2</sup>) and that all bars have a circular cross section of diameter 0.5 cm, i.e., that  $a_i = 0.0025^2 \pi$  m<sup>2</sup>. Furthermore, we assume that the lengths of the horizontal and vertical bars are 2m and 1m, respectively. (Obviously, the diagonal bars then have length  $\sqrt{2^2 + 1^2}$  m). Given this information, we can compute  $D$  from (2.41). We hang a mass of 100 kg from node 3. Thus, the applied force is  $F = 100g$ , where  $g = 9.80665$  m/sec<sup>2</sup> is the standard acceleration of gravity. (Of course, we are somewhat inconsistent here because we neglect forces due to acceleration of gravity acting on the bars themselves.) For these data the



**Figure 2.8:** Undeformed and deformed truss.

solution  $u$ , rounded to four digits, of the system (2.43) with  $B$  and  $f$  given in (2.46) is

$$\begin{pmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} = 10^{-3} \begin{pmatrix} 0.499 \\ 2.394 \\ -0.525 \\ 2.519 \end{pmatrix} \quad [\text{m}].$$

The plot of the corresponding deformed truss can not be distinguished from the undeformed one.

To compute the displacements  $u$ , we have to solve the system (2.43). Note that the decomposition  $K = BDB^T$  looks like the  $LDL^T$  decomposition of  $K$ , but it is not the  $LDL^T$  decomposition because  $B \in \mathbb{R}^{(2n-s) \times m}$ , ( $s$  is the number of fixed displacements), is usually not square and not a triangular matrix. The stiffness matrix  $K = BDB^T$  can be shown to be symmetric and positive semi-definite. In fact, since  $D$  is a diagonal matrix

$$K^T = (BDB^T)^T = (B^T)^T D^T B^T = BDB^T = K$$

and, since all diagonal entries of  $D$  are positive,

$$x^T Kx = x^T BDB^T x = \underbrace{(B^T x)^T}_{=z} D \underbrace{(B^T x)}_{=z} = \sum_{i=1}^m d_i z_i^2 \geq 0$$

for all  $x \in \mathbb{R}^{2n-s}$ . The stiffness matrix  $K = BDB^T$  is symmetric and positive definite if and only if the rows of  $B \in \mathbb{R}^{(2n-s) \times m}$  are linearly independent. If the rows of  $B \in \mathbb{R}^{(2n-s) \times m}$  are linearly independent, then  $x \neq 0$  implies that  $z = B^T x \neq 0$ . Thus, if the rows of  $B \in \mathbb{R}^{(2n-s) \times m}$  are linearly independent, we can use the  $LDL^T$  or the Cholesky decomposition of  $K$  to solve (2.43).

We close this section with a note on the implementation. Information concerning the bar topology is stored in two arrays. The first array, `bars`, is a  $m \times 2$  integer array. The  $i$ th row of `bars` contains the following information:

- `bars(i,1)`: index of the left lower node of bar  $i$
- `bars(i,2)`: index of the right upper node of bar  $i$

The second array, **nodes**, is a  $n \times 2$  real array. The  $j$ th row of **nodes** contains the following information:

$$\begin{aligned} \text{nodes}(j,1): & \quad x\text{-coordinate of node } j \\ \text{nodes}(j,2): & \quad y\text{-coordinate of node } j \end{aligned}$$

The angle  $\theta_i$  for the  $i$ th bar can be computed from the information stored in **bars** and **nodes**. In fact<sup>2</sup>, if

$$\tilde{\theta}_i = \arctan \left( \frac{\text{nodes}(\text{bars}(i,2),2) - \text{nodes}(\text{bars}(i,1),2)}{\text{nodes}(\text{bars}(i,2),1) - \text{nodes}(\text{bars}(i,1),1)} \right),$$

then

$$\theta_i = \begin{cases} -\tilde{\theta}_i & \text{if } \tilde{\theta}_i \geq 0, \\ -(\pi + \tilde{\theta}_i) & \text{otherwise,} \end{cases}$$

provided  $\text{nodes}(\text{bars}(i,2),1) - \text{nodes}(\text{bars}(i,1),1) \neq 0$ . If  $\text{nodes}(\text{bars}(i,2),1) - \text{nodes}(\text{bars}(i,1),1) = 0$ , then

$$\theta_i = -\pi/2.$$

The length  $\ell_i$  of the  $i$ th bar can also be computed from **bars** and **nodes**. In addition to the two arrays **bars** and **nodes**, we need another  $m \times 2$  real array that contains the Young's modulus  $E_i$  and the cross sectional area  $a_i$  for each bar.

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<sup>2</sup>Keep in mind our sign convention for the angle  $\theta_i$ .