HW 5 - Problems 1 and 2

1. Suppose $K: [0,1] \times [0,1] \Rightarrow \mathbb{R}$ is continuous. Define $T: C^0[0,1] \to C^0[0,1]$ by

$$Tf(x) = \int_0^1 dy\, K(x,y) f(y)$$

Show that T maps bounded sets to precompact sets.

<u>Pf.</u> Suffices to show: Suppose $\{f_n\} \subset C^0[0,1]$ is a bdd sequence. Then $\{Tf_n\}$ has a convergent subsequence. [Def^{n} of precompact, given in class]

Thus assume $M \geq 0$ st. $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$, $x \in [0,1]$. [$C^0[0,1]$ is metric space with sup norm.]
$[0,1] \times [0,1]$ compact $\Rightarrow$ K attains its ~~sup~~, in particular is bdd [Thm 4.16]: for some $B \geq 0$ $|K(x,y)| \leq B$ for all $(x,y) \in [0,1] \times [0,1]$.

So

$$|Tf_n(x)| = \left| \int_0^1 dy\, K(x,y) f_n(y) \right|$$

$$\leq \int_0^1 dy\, |K(x,y)| |f_n(y)| \quad \text{[Thm 6.13]}$$

$$\leq \int_0^1 dy\, B\, M \leq B \cdot M$$

$\Rightarrow$ $\{Tf_n\}$ is <u>pointwise bounded</u> (in fact, <u>uniformly</u>!

Because K is cont. and $[0,1] \times [0,1]$ is compact, K is unif. <u>cont.</u> [Thm 4.19]: given $\varepsilon > 0$, there is $\delta > 0$ st. $\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} < \delta \Rightarrow |K(x_1,y_1) - K(x_2,y_2)| < \varepsilon/M$.

$\therefore$ if $|x-z| < \delta$,

$$|Tf_n(x) - Tf_n(z)| = \left|\int_0^1 dy\,(K(x,y) - K(z,y))\, f_n(y)\right|$$

$$\leq \int_0^1 dy\, |K(x,y) - K(z,y)|\; |f_n(y)|$$

$$\leq \int_0^1 dy\, \frac{\varepsilon}{M} \cdot M$$

$$= \varepsilon.$$

Thus $\{Tf_n\}$ is equicontinuous.

Since $[0,1]$ is compact, Thm 7.25 (Ascoli's) $\Rightarrow$ $\{Tf_n\}$ contains a unif. convgt. subseq.

2. (7.22) According to Prob. 6.12, which you are allowed to use, given $n \in \mathbb{N}$ there exists $g \in C^0[a,b]$ for which

$$\int_a^b |f - g_n|^2\, d\alpha < \frac{1}{4n}$$

According to the Weierstrass thm (7-26), there exists a polynomial P_n for which

$$|g_n(x) - P_n(x)| < \frac{1}{\sqrt{4n(\alpha(b)-\alpha(a))}}, \quad x \in [a,b]$$

whence

$$\int_a^b |g_n - P_n|^2\, d\alpha < \frac{1}{4n} \qquad \text{(Thm 6.12(d))}$$