Effect of correlations on information – example calculation

We are interested in the impact of uniform correlations of magnitude c on the Fisher information in a population, assuming correlated Poisson variability.

Correlation matrix (N-dimensional):

$$\mathbf{C} = \underbrace{\begin{pmatrix} 1 & c & c \\ c & 1 & c \\ c & c & 1 \end{pmatrix}}_{N \text{ dimensions}} = (1-c)\mathbf{I} + c\mathbf{1}\mathbf{1}^{\mathrm{T}},$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{1} = (1,\ldots,1)^{\mathrm{T}}$.

Fisher information in terms of the covariance matrix $\mathbf{\Sigma}$:

$$I(s) = \mathbf{f}'^{\mathrm{T}}(s)\,\mathbf{\Sigma}^{-1}(s)\,\mathbf{f}'(s)$$

Relation between covariance matrix and correlation matrix:

$C_{ij} = \dfrac{\Sigma_{ij}}{\sigma_i \sigma_j} = \dfrac{\Sigma_{ij}}{\sqrt{f_i f_j}}$, since we assume Poisson variability (variance = mean). This can be written as

$$\mathbf{\Sigma} = \begin{pmatrix} \sqrt{f_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sqrt{f_N} \end{pmatrix} \mathbf{C} \begin{pmatrix} \sqrt{f_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sqrt{f_N} \end{pmatrix}$$

and therefore Fisher information is

$$I = \mathbf{f}'^{\mathrm{T}}\mathbf{\Sigma}^{-1}\mathbf{f}' = \mathbf{f}'^{\mathrm{T}} \begin{pmatrix} f_1^{-\frac{1}{2}} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & f_N^{-\frac{1}{2}} \end{pmatrix} \mathbf{C}^{-1} \begin{pmatrix} f_1^{-\frac{1}{2}} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & f_N^{-\frac{1}{2}} \end{pmatrix} \mathbf{f}' = \begin{pmatrix} \frac{f_1'}{\sqrt{f_1}} \\ \vdots \\ \frac{f_N'}{\sqrt{f_N}} \end{pmatrix}^{\mathrm{T}} \mathbf{C}^{-1} \begin{pmatrix} \frac{f_1'}{\sqrt{f_1}} \\ \vdots \\ \frac{f_N'}{\sqrt{f_N}} \end{pmatrix} \equiv \boldsymbol{\mu}^{\mathrm{T}}\mathbf{C}^{-1}\boldsymbol{\mu}$$

where we denote $\boldsymbol{\mu} = \left(\dfrac{f_1'}{\sqrt{f_1}}, \ldots, \dfrac{f_N'}{\sqrt{f_N}} \right)^{\mathrm{T}}$.

The vector $\mathbf{1}$ is an eigenvector of $\mathbf{C}$ with eigenvalue $1 + (N-1)c$.

Now observe that C is symmetric. Therefore, its eigenvectors are all orthogonal and right and left eigenvectors are the same. Any other eigenvector than **1** must be orthogonal to **1**. Conversely, all vectors that are orthogonal to **1** are eigenvectors:

$$\mathbf{C}\boldsymbol{\chi} = (1-c)\mathbf{I}\boldsymbol{\chi} + c\mathbf{1}\mathbf{1}^{\mathrm{T}}\boldsymbol{\chi} = (1-c)\boldsymbol{\chi}$$

All these eigenvectors have eigenvalue 1-c. The eigenvalue 1-c has eigenspace with dimension N-1 and this space is orthogonal to the the vector **1.** All vectors orthogonal to **1** are eigenvectors and any orthogonal set of vectors which span the eigenspace can be used.

If the tuning curves are symmetric, then $\boldsymbol{\mu}$ is orthogonal to **1**. Therefore, $\boldsymbol{\mu}$ is an eigenvector with eigenvalue 1-c. We normalize it to obtain $\boldsymbol{\chi}_0 = \frac{\boldsymbol{\mu}}{\sqrt{\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\mu}}}$.

We can diagonalize **C** as $\mathbf{C} = \mathbf{X}\boldsymbol{\Lambda}\mathbf{X}^{-1}$, with **X** a matrix of orthonormal eigenvectors and $\boldsymbol{\Lambda}$ the diagonal matrix of eigenvalues. Because the eigenvectors are orthonormal, $\mathbf{X}^{\mathrm{T}} = \mathbf{X}^{-1}$ and $\mathbf{C} = \mathbf{X}\boldsymbol{\Lambda}\mathbf{X}^{\mathrm{T}}$, or in other words $\mathbf{C} = \sum_i \lambda_i \boldsymbol{\chi}_i \boldsymbol{\chi}_i^{\mathrm{T}}$. Similarly, $\mathbf{C}^{-1} = \sum_i \lambda_i^{-1} \boldsymbol{\chi}_i \boldsymbol{\chi}_i^{\mathrm{T}}$.

Information is therefore

$$I = \boldsymbol{\mu}^{\mathrm{T}}\mathbf{C}^{-1}\boldsymbol{\mu} = \boldsymbol{\mu}^{\mathrm{T}}\sum_i \lambda_i \boldsymbol{\chi}_i \boldsymbol{\chi}_i^{\mathrm{T}}\boldsymbol{\mu} = \frac{1}{1-c}\boldsymbol{\mu}^{\mathrm{T}}\frac{\boldsymbol{\mu}\boldsymbol{\mu}^{\mathrm{T}}}{\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\mu}}\boldsymbol{\mu} = \frac{1}{1-c}\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\mu}$$

For c = 0.2, the minimal variance of estimates will be 20% smaller than if c=0.

Note also that

$$I_{\mathrm{shuffled}}(s) = \sum_i \frac{\left(f_i'(s)\right)^2}{f_i(s)} = \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\mu}$$

This is the amount of information contained in a population in which individual neurons have the same response distributions as above, but are uncorrelated. (This is achieved by shuffling the trials.)

Furthermore,

$$I_{\mathrm{diag}}(s) = \frac{\left(\mathbf{f}'^{\mathrm{T}}\boldsymbol{\Sigma}_{\mathrm{diag}}^{-1}\mathbf{f}'\right)^2}{\mathbf{f}'^{\mathrm{T}}\boldsymbol{\Sigma}_{\mathrm{diag}}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}_{\mathrm{diag}}^{-1}\mathbf{f}'} = \frac{\left(\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\mu}\right)^2}{\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\Sigma}_{\mathrm{diag}}^{-1/2}\boldsymbol{\Sigma}\boldsymbol{\Sigma}_{\mathrm{diag}}^{-1/2}\boldsymbol{\mu}} = \frac{\left(\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\mu}\right)^2}{\boldsymbol{\mu}^{\mathrm{T}}\mathbf{C}\boldsymbol{\mu}} = \frac{\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\mu}}{1-c} = I(s)$$

This is the amount of information that would be extracted from the correlated population when using a suboptimal decoder that is optimal for the shuffled (decorrelated) responses.