Solution to exercise 2

The steady-state cable equation is:

$$\lambda \frac{d^2V}{dx^2} = V.$$

We use the Ansatz, $V(x) = c_1 e^{x/\lambda} + c_2 e^{-x/\lambda}$ to obtain:

$$\frac{dV}{dx} = \frac{c_1}{\lambda} e^{x/\lambda} - \frac{c_2}{\lambda} e^{-x/\lambda}, \quad \text{so that} \quad \frac{d^2V}{dx^2} = \frac{c_1}{\lambda^2} e^{x/\lambda} + \frac{c_2}{\lambda} e^{-x/\lambda} = \frac{1}{\lambda^2} V(x).$$

From the boundary conditions,

$$\left.\frac{dV}{dx}\right|_{x=0} = -\gamma, \quad \text{and} \quad \left.\frac{dV}{dx}\right|_{x=l} = 0$$

we obtain:

$$\frac{c_1}{\lambda} - \frac{c_2}{\lambda} = -\gamma \quad \text{and} \quad \frac{c_1}{\lambda} e^{l/\lambda} - \frac{c_2}{\lambda} e^{-l/\lambda} = 0.$$

Rearranging gives,

$$c_1 - c_2 = -\gamma\lambda \quad \text{and} \quad c_2 = c_1 e^{2l/\lambda}.$$

Combining these two equations, we obtain first

$$c_1 = \frac{-\gamma\lambda}{1 - e^{2l/\lambda}} \quad \text{and then} \quad c_2 = \frac{-e^{2l/\lambda}\gamma\lambda}{1 - e^{2l/\lambda}}.$$

We now plug the values of c_1 and c_2 in the Ansatz,

$$\begin{aligned} V(x) &= \frac{-\gamma\lambda}{1 - e^{2l/\lambda}} e^{x/\lambda} - \frac{e^{2l/\lambda}\gamma\lambda}{1 - e^{2l/\lambda}} e^{-x/\lambda}, \\ &= \frac{-\gamma\lambda e^{-l/\lambda}}{e^{-l/\lambda}(1 - e^{2l/\lambda})} (e^{x/\lambda} + e^{2l/\lambda} e^{-x/\lambda}), \\ &= \frac{-\gamma\lambda}{e^{-l/\lambda} - e^{l/\lambda}} (e^{(x-l)/\lambda} + e^{-(x-l)/\lambda}), \\ &= \frac{\gamma\lambda}{\sinh l/\lambda} (\cosh(l-x)/\lambda). \end{aligned}$$

We can now compute $V(0) = \gamma\lambda \frac{\cosh l/\lambda}{\sinh l/\lambda}$ and rewrite

$$V(x) = V(0) \frac{\cosh(l-x)/\lambda}{\cosh l/\lambda}.$$

To take the limit $l \to \infty$, we rewrite

$$V(x) = V(0)\frac{e^{l/\lambda}}{e^{l/\lambda}}\frac{(e^{-x/\lambda} + e^{-2l/\lambda}e^{-x/\lambda})}{(1 + e^{-2l/\lambda})}.$$

Since $e^{-2l/\lambda} \to 0$ as $l \to \infty$, the 2nd additive terms in both the numerator and denominator of this last equation tend to zero. Thus,

$$V(x) = V(0)e^{-x/\lambda}.$$

Example of cable constant. From the values given in the exercise statement and the formula for the space constant in the lecture notes, we obtain

$$\lambda = \sqrt{\frac{100,000\,\Omega\,\mathrm{cm}^2}{100\,\Omega\,\mathrm{cm}} \cdot \frac{10^{-4}\,\mathrm{cm}}{4}} = 0.1581\,\mathrm{cm} = 1581\,\mu\mathrm{m}.$$

Since the dendrites of pyramidal cells are of smaller length ($\sim 300\mu m$) than the space constant, the infinite cable approximation is not appropriate.