CAAM 440 · APPLIED MATRIX ANALYSIS

Pledged Problem Set 3

Posted Friday 13 March 2012. [Problem 1 corrected 15 March 2012.] Due Friday 20 April 2012.

Complete any four problems, 25 points each. *You are welcome to complete more problems if you like, but specify which four you want to be graded.*

Rules: On this pledged problem set, you are welcome to use books, web pages, MATLAB, Mathematica, etc. (Further restrictions may apply to future pledged problem sets.) You are ***not allowed*** to discuss the problems with anyone aside from the instructor. Please write out and sign the traditional pledge on your assignment. **Pledged problem sets will not be accepted late unless you have made a previous arrangement with the instructor.**

1. Recall the Jordan block

$$\mathbf{J}_n = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \in \mathbb{C}^{n\times n}.$$

In class, we saw that the vector

$$\mathbf{v}_n = \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \\ z^{n-1} \end{pmatrix} \in \mathbb{C}^n$$

was an approximate eigenvector corresponding to the approximate eigenvalue $z \in \mathbb{C}$, in the sense that $\|\mathbf{J}_n\mathbf{v}_n - z\mathbf{v}_n\|$ is small provided $|z| < 1$ (and gets exponentially smaller as n increases.

Now consider the *Toeplitz matrix*

$$\mathbf{A}_n = \begin{pmatrix} a_0 & a_1 & a_2 & & \\ & a_0 & a_1 & \ddots & \\ & & \ddots & \ddots & a_2 \\ & & & a_0 & a_1 \\ & & & & a_0 \end{pmatrix} \in \mathbb{C}^{n\times n}.$$

which has the constant entries on diagonals, and define the function $a(z) = a_0 + a_1 z + a_2 z^2$.

(a) For the vector $\mathbf{v}_n$ specified above, compute $\|\mathbf{A}_n\mathbf{v}_n - a(z)\mathbf{v}_n\|$.

(b) If $|z| < 1$, how does $\|\mathbf{A}_n\mathbf{v}_n - a(z)\mathbf{v}_n\|$ behave as $n \to \infty$?
What does this imply about $\|(a(z)\mathbf{I} - \mathbf{A}_n)^{-1}\|$ as $n \to \infty$, for $|z| < 1$?

(c) Take $a_0 = 0$, $a_1 = 2$, and $a_2 = 1$. Plot the curve

$$a(\mathbb{T}) := \{a(\mathrm{e}^{i\theta}) : \theta \in [0, 2\pi)\}$$

and indicate the area where $|z| < 1$.
You can plot this curve in MATLAB via

```
T = exp(linspace(0,2i*pi,500));
aT = a0*T.^0 + a1*T.^1 + a2*T.^2;
plot(real(aT),imag(aT),'-');
```

(d) For dimensions $n = 50, 100, 200$, construct $\mathbf{A}_n$ and plot the eigenvalues of $\mathbf{A}_n + \mathbf{E}$, where $\mathbf{E}$ is a random matrix of norm 10^{-3}:

```
E = randn(n); E = 1e-3*E/norm(E);
```

(e) [optional] The same theory extends (with a different form of $\mathbf{v}$) for banded Toeplitz matrices,

$$\mathbf{A}_n = \begin{bmatrix} a_0 & a_1 & \cdots & a_p & & \\ a_{-1} & a_0 & \ddots & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & a_p \\ a_{-m} & \ddots & \ddots & \ddots & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & a_1 \\ & & a_{-m} & \cdots & a_{-1} & a_0 \end{bmatrix} \in \mathbb{C}^{n \times n}$$

with

$$a(z) = \sum_{j=-m}^{p} a_j z^j.$$

Repeat parts (c) and (d) for the values

$$a_{-1} = -1, \quad a_0 = a_1 = a_2 = a_3 = 1,$$

which gives the *Grcar matrix.*

2. (a) Suppose that $\mathbf{A} \in \mathbb{R}^{n \times n}$ is positive, $\mathbf{A} > \mathbf{0}$. Prove that

$$\min_{1 \le j \le n} \sum_{k=1}^{n} a_{j,k} \le \rho(\mathbf{A}) \le \max_{1 \le j \le n} \sum_{k=1}^{n} a_{j,k},$$

where $\rho(\mathbf{A})$ denotes the spectral radius of $\mathbf{A}$. [Meyer]

(b) Illustrate these bounds for matrices with uniform random entries in $[0, 1/n]$, i.e., `A = rand(n)/n`, for dimensions `n` $= 16, 64, 256$ and 1024.

3. (a) Compute the spectral radius and Perron vector (the eigenvector $\mathbf{x} > \mathbf{0}$ associated with $\rho(\mathbf{A})$ having $\sum_{j=1}^{n} x_j = 1$) for the matrix

$$\mathbf{A} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

with $\alpha + \beta = 1$ and $\alpha, \beta \in [0, 1]$. [Meyer]

(b) Interpret the matrix $\mathbf{A}$ in part (a) as the transition matrix for a two-state Markov chain. What is $\mathbf{A}^\infty := \lim_{k\to\infty} \mathbf{A}^k$? Describe the limiting steady state, $\mathbf{p}_\infty^* := \mathbf{p}_0^* \mathbf{A}^\infty$. At what rate is this limit reached? Does this depend on the values of α and β?

4. Suppose $\mathbf{A}$ is an adjacency matrix for an undirected graph with n vertices, i.e.,

$$a_{j,k} = \begin{cases} 1, & \text{an edge connects vertex } j \text{ to vertex } k; \\ 0, & \text{otherwise.} \end{cases}$$

In particular, the diagonal of $\mathbf{A}$ is zero.

In class we noted that the number of paths of length m from node j to node k equals the (j, k) entry of $\mathbf{A}^m$. Inspired by this method of path-counting, Estrada and Rodríguez-Velázquez (2005) proposed a way to gauge the importance of each vertex as a weighted sum of the number of paths from a given vertex back to itself. In particular, the *subgraph centrality* (or *Estrada index*) of vertex j is given by

$$(\mathrm{e}^{\mathbf{A}})_{j,j}.$$

The higher this value, the more central a vertex is regarded. Estrada and Hatano (2007) extend this notion to measure the *communicability* of vertices j and k as

$$(\mathrm{e}^{\mathbf{A}})_{j,k}.$$

(This work is surveyed and extended in a paper by Estrada and D. J. Higham, *SIAM Review*, 2010.)

We now wish to apply these ideas to real data. Analyzing a variety of media sources, Valdis Krebs has produced a graph of the 9/11 terrorist network, including both the hijackers and their accomplices; see `http://www.orgnet.com/hijackers.html` for details. We have written a routine `sept11.m`, that creates an adjacency matrix for Krebs's graph, given an arbitrary vertex numbering. The graph contains 69 vertices, 19 of which correspond to the hijackers. (These nodes are identified at the top of `sept11.m`, with the pilots singled out.)

(a) Compute the subgraph centrality for all the vertices, and present a table showing the results for the top thirty vertices. Each row of this table should correspond to a vertex, and include:

  i. subgraph centrality rank (in order, 1,..., 30);
  ii. vertex number, j;
  iii. subgraph centrality value, $(\mathrm{e}^{\mathbf{A}})_{j,j}$;
  iv. number of edges connected to vertex j;

(b) In your table, clearly identify the four pilots (nodes 36, 44, 50, 65; see `sept11.m` for details).

(c) We might alternatively have ranked the vertices by their edge counts, rather than subgraph centrality. Does your table reveal any anomalous vertices that are ranked more highly than their edge counts would suggest, or vertices with large edge counts that have relatively low centrality scores?

(d) Now use `imagesc` in MATLAB (or similar) to visualize the magnitude of the entries in $\mathrm{e}^{\mathbf{A}}$. This allows you to visually inspect the *communicability* of all vertices with one another at once. Designate the rows and columns corresponding to the hijackers of each plane. You may do this by hand, or using some slick command along the lines of, e.g.,

```
fill([0 70 70 0],[31.5 31.5 36.5 36.5],'w','facealpha',.3,'edgealpha',.3)
```

as you like.

(e) Interpret your plot in part (d). Were the different crews of hijackers highly connected with one another? Do you notice anything about the crew of United Flight 93, which crashed in Pennsylvania as a result of passenger intervention?

5. Suppose that $\mathbf{A} \in \mathbb{R}^{n\times n}$ with $\mathbf{A} > \mathbf{0}$ for parts (a)–(c).

(a) Argue, by way of similarity transformation, that the result of Problem 2(a) can be adapted to

$$\min_{1\le j\le n} \frac{1}{x_j}\sum_{k=1}^{n} a_{j,k}x_k \le \rho(\mathbf{A}) \le \max_{1\le j\le n} \frac{1}{x_j}\sum_{k=1}^{n} a_{j,k}x_k,$$

for *any* $\mathbf{x} > \mathbf{0}$. (Note that $\mathbf{x}$ need not be the Perron vector.)

(b) Show that $\rho(\mathbf{A})$ is the only eigenvalue of $\mathbf{A}$ that has an eigenvector with all entries positive.

(c) Prove the "max min" characterization of $\rho(\mathbf{A})$ for positive matrices:

$$\rho(\mathbf{A}) = \max_{\mathbf{x}>\mathbf{0}} \min_{1\le j\le n} \frac{1}{x_j}\sum_{k=1}^{n} a_{j,k}x_k = \min_{\mathbf{x}>\mathbf{0}} \max_{1\le j\le n} \frac{1}{x_j}\sum_{k=1}^{n} a_{j,k}x_k.$$

(d) Show by example that the result of part (b) need not hold if we only require $\mathbf{A}$ and its eigenvector to be *nonnegative*. That is, build $\mathbf{A} \geq \mathbf{0}$ with eigenvalue $\lambda \neq \rho(\mathbf{A})$ that has a nonnegative eigenvector.

[Horn and Johnson]

6. (a) Suppose that $\mathbf{A} \in \mathbb{R}^{n\times n}$ is nonnegative, $\mathbf{A} \geq \mathbf{0}$. Prove that conditions (i) and (ii) below are equivalent. [Varga]

   (i) $\alpha > \rho(\mathbf{A})$.

   (ii) $\alpha\mathbf{I} - \mathbf{A}$ is invertible and $(\alpha\mathbf{I} - \mathbf{A})^{-1} \geq \mathbf{0}$.

   (b) The finite difference discretization of the differential equation $-u''(x) = f(x)$ for $x \in [0,1]$ with $u(0) = u(1) = 0$ on a grid of points $x_0, \ldots, x_{n+1}$ with $x_j = jh$ for $h = 1/(n+1)$ leads to a matrix equation of the form

$$h^{-2}\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}.$$

   Denote this equation as $\mathbf{Au} = \mathbf{f}$ (note that $\mathbf{A}$ includes the h^{-2} factor). Use the fact that

$$\sigma(\mathbf{A}) = \{h^{-2}(2 + 2\cos(k\pi/(n+1))) : k = 1, \ldots, n\}$$

   (which you do not need to prove) and the result in (a) to show that $\mathbf{A}^{-1} > \mathbf{0}$.
   (This $\mathbf{A}$ is an example of an *M-matrix*, an important class of matrices for many applications.)