Problem 1 (30 points)

Any two norms \(\|\cdot\|_a\) and \(\|\cdot\|_b\) defined on a finite-dimensional space (assumed to be \(\mathbb{R}^n\)) are equivalent in the sense that there exist constants \(c, C > 0\) such that for every \(x\),

\[
c \|x\|_b \leq \|x\|_a \leq C \|x\|_b. \tag{1}
\]

(a) Prove that any norm \(\|\cdot\|_a\) is a continuous function under itself. Consider here the \(\epsilon-\delta\) definition of limit and the triangular inequality of the definition of \(\|\cdot\|_a\).

(b) Prove (1) when \(\|\cdot\|_b = \|\cdot\|_\infty\). Hint: First apply the Weirstrass theorem to \(\|\cdot\|_a\) (continuous) on \(\Omega = \{x \in \mathbb{R}^n : \|x\|_\infty = 1\}\) (closed and bounded).

OPTIONAL: Prove that \(\|\cdot\|_a\) is indeed continuous under \(\|\cdot\|_\infty\).

(c) Prove that

\[
\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n} \|x\|_\infty. \tag{2}
\]

(d) For \(n = 2\), using b) and drawing the level curves of the \(\infty\)-norm, arrive at the same constants as in c).

(e) Show that the bound (2) is sharp: Find all vectors where \(\|x\|_\infty = \|x\|_2\). Similarly, find all vectors where \(\|x\|_2 = \sqrt{n} \|x\|_\infty\).

(f) It follows that the 2-matrix norm and \(\infty\)-matrix norm are equivalent. In fact, if \(A \in \mathbb{R}^{m \times n}\),

\[
\frac{1}{\sqrt{n}} \|A\|_\infty \leq \|A\|_2 \leq \sqrt{m} \|A\|_\infty.
\]

Find two nonzero matrices (with general \(m \) and \(n\)) for which each inequality above is an equality.
Problem 2 (10 points)

Let $A \in \mathbb{C}^{m \times n}$. Let $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ be unitary matrices.

(a) Prove $\|UA\|_F = \|A\|_F$.

(b) Using a), prove also $\|AV\|_F = \|A\|_F$.

Problem 3 (20 points) \hspace{1em} Let $x \in \mathbb{C}^n$. Consider the matrix $\mathbb{C}^{n \times n}$ obtained from the identity changing only the components in positions $i, i, j, i$ and $j, j$.

$$G = \begin{bmatrix}
1 & & & \\
& \ddots & & \\
& & 1 & \\
& & \bar{c} & \cdots & \bar{s} \\
& & \vdots & \ddots & \vdots \\
& & -s & \cdots & c \\
& & & & 1 \\
& & & \ddots & \\
& & & & 1
\end{bmatrix},$$

where 

$$c = \frac{x_i}{\sqrt{|x_i|^2 + |x_j|^2}} \quad \text{and} \quad s = \frac{x_j}{\sqrt{|x_i|^2 + |x_j|^2}}.$$ 

The indices $i$ and $j$ verify $i \neq j$ and $i, j \in \{1, \ldots, n\}$. This matrix is called the Givens rotation matrix associate with the vector $x$.

(a) Show that the $j$-th entry of $Gx$ is zero.

(b) Prove that $G$ is unitary.

(c) Given $z \in \mathbb{C}^n$, show how to determine a unitary matrix $U$ using $n - 1$ Givens rotations such that $Uz$ is a multiple of the first column of the identity.

Problem 4 (40 points)

In this problem you will write a code to compute the minimum norm solution of a underdetermined system of linear equations, written in the form $Ax = b$ with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $m < n$. For that you can assume that $A$ has rank $m$ and that the minimum norm solution is given by $x_* = A^\top (AA^\top)^{-1}b$.

(a) Express $x_*$ in terms of the factors of the QR decomposition of $A^\top$. 

\[ \begin{array}{l}
\text{Problem 2 (10 points)} \\
\text{Problem 3 (20 points)} \\
\text{Problem 4 (40 points)}
\end{array} \]

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\text{Problem 2 (10 points)} \\
\text{Problem 3 (20 points)} \\
\text{Problem 4 (40 points)}
\end{aligned} \]
(b) Your code should include the following functions:

- \([Q,R] = QRHouseholder(B)\). Input: real \(m \times n\) matrix \(B\). Output: \(m \times m\) unitary \(Q\) and \(n \times n\) upper triangular \(R\) such that \(QR = B\), or an error if \(B\) has not rank \(n\). Use Householder reflections to calculate \(Q\) and \(R\).

- \(s = LowerBackSolve(R,v)\). Input: lower triangular \(n \times n\) matrix \(R\) and \(n \times 1\) vector \(v\). Output: \(n \times 1\) vector \(s = R^{-1}v\), or an error if \(R\) is singular.

- \([x,\text{norm}x] = MinNorm(A,b)\). Input: real \(m \times n\) matrix \(A\) and \(m \times 1\) vector \(b\). Output: \(n \times 1\) vector and corresponding norm \(\text{norm}x\), or an error if \(A\) has not rank \(m\). Use \(QRHouseholder\) and \(LowerBackSolve\).

You can use all scalar and vector operations, and matrix multiplication. Of course, you cannot use \(\backslash\) or \(/\) or any other command that can solve linear equations.

Test your code on simple two- and three dimensional examples.

(c) **OPTIONAL:** Can you still use the \(QR\) factorization of \(A^T\) together with \(A^T(AA^T)^{-1}b\) to compute a minimum norm solution when \(A^T\) does not have rank \(m\)?