CAAM 453: Numerical Analysis I

Homework Assignment 1: Norms; QR.
Due: September 9, 2016

Justify well all your answers. Note: Turn in all MATLAB code that you have written and turn in all output generated by your MATLAB functions/scripts. MATLAB functions/scripts must be commented, output must be formatted nicely, and plots must be labeled. You may freely use any MATLAB functions from this year’s CAAM 453 edition. If you do so, you do not need to include their code in your writeup unless you have modified them.

Problem 1: Vector and matrix norms (30 points)

Any two norms $\| \cdot \|_a$ and $\| \cdot \|_b$ defined on a finite-dimensional space (assumed to be $\mathbb{R}^n$) are equivalent in the sense that there exist constants $c, C > 0$ such that for every $x$,

$$c \| x \|_b \leq \| x \|_a \leq C \| x \|_b.$$  \hfill (1)

(a) Prove that any norm $\| \cdot \|$ is a continuous function under itself. Consider here the $\epsilon$–$\delta$ definition of limit and the triangular inequality of the definition of $\| \cdot \|$.

(b) Prove (1) when $\| \cdot \|_b = \| \cdot \|_\infty$. Hint: First apply the Weierstrass theorem to $\| \cdot \|_a$ (continuous) on $\Omega = \{ x \in \mathbb{R}^n : \| x \|_\infty = 1 \}$ (closed and bounded).

OPTIONAL: Prove that $\| \cdot \|_a$ is indeed continuous under $\| \cdot \|_\infty$.

(c) Prove that

$$\| x \|_\infty \leq \| x \|_2 \leq \sqrt{n} \| x \|_\infty.$$  \hfill (2)

(d) For $n = 2$, using b) and drawing the level curves of the 2–norm, arrive at the same constants as in c).

(e) Show that the bound (2) is sharp: Find all vectors where $\| x \|_\infty = \| x \|_2$. Similarly, find all vectors where $\| x \|_2 = \sqrt{n} \| x \|_\infty$.

(f) It follows that the 2-matrix norm and $\infty$-matrix norm are equivalent. In fact, if $A \in \mathbb{R}^{m \times n}$,

$$\frac{1}{\sqrt{n}} \| A \|_\infty \leq \| A \|_2 \leq \sqrt{m} \| A \|_\infty.$$  

Find two nonzero matrices (with general $m$ and $n$), one for which each inequality, so that the corresponding inequality is an equality.
Problem 2: Matrix norms and unitary transformations (10 points)

Let \( A \in \mathbb{C}^{m \times n} \). Let \( U \in \mathbb{C}^{m \times m} \) and \( V \in \mathbb{C}^{n \times n} \) be unitary matrices.

(a) Prove \( \|UA\|_F = \|A\|_F \).

(b) Using a), prove also \( \|AV\|_F = \|A\|_F \).

Problem 3: More about unitary transformations (Givens rotations) (20 points)

Let \( x \in \mathbb{C}^n \). Consider the matrix \( \mathbb{C}^{n \times n} \) obtained from the identity changing only the components in positions \( i, i, \quad i, j, \quad j, i \) and \( j, j \),

\[
G = \begin{bmatrix}
1 & \cdots & \cdots & \cdots & \cdots \\
. & 1 & \ddots & \ddots & \ddots \\
& \ddots & \ddots & \ddots & \ddots \\
m & \ddots & \ddots & \ddots & 1 \\
& & & & 1
\end{bmatrix},
\]

where
\[
c = \frac{x_i}{\sqrt{|x_i|^2 + |x_j|^2}} \quad \text{and} \quad s = \frac{x_j}{\sqrt{|x_i|^2 + |x_j|^2}}.
\]

The indices \( i \) and \( j \) verify \( i < j \) and \( i, j \in \{1, \ldots, n\} \). This matrix is called the Givens rotation matrix associate with the vector \( x \).

(a) Show that the \( j \)-th entry of \( Gx \) is zero.

(b) Prove that \( G \) is unitary.

(c) Given \( z \in \mathbb{C}^n \), show how to determine a unitary matrix \( U \) using \( n - 1 \) Givens rotations such that \( Uz \) is a multiple of the first column of the identity.

Problem 4 Use of QR decomposition (40 points)

In this problem you will write a code to compute the minimum norm solution of an underdetermined system of linear equations, written in the form \( Ax = b \) with \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \), and \( m < n \). For that you can assume that \( A \) has rank \( m \) and that the minimum norm solution is given by \( x_\ast = A^\top(AA^\top)^{-1}b \).

(a) Express \( x_\ast \) in terms of the factors of the QR decomposition of \( A^\top \).
(b) Your code should include the following functions:

- \([Q,R] = \text{QRHouseholder}(B)\). Input: real \(m \times n\) matrix \(B\). Output: \(m \times m\) unitary \(Q\) and \(m \times n\) upper triangular \(R\) such that \(QR = B\), or an error if \(B\) has not rank \(n\). Use Householder reflections to calculate \(Q\) and \(R\).

- \(s = \text{LowerBackSolve}(R,v)\). Input: lower triangular \(n \times n\) matrix \(R\) and \(n \times 1\) vector \(v\). Output: \(n \times 1\) vector \(s = R^{-1}v\), or an error if \(R\) is singular.

- \([x,\text{normx}] = \text{MinNorm}(A,b)\). Input: real \(m \times n\) matrix \(A\) and \(m \times 1\) vector \(b\). Output: \(n \times 1\) vector and corresponding norm \(\text{normx}\), or an error if \(A\) has not rank \(m\). Use QRHouseholder and LowerBackSolve.

You can use all scalar and vector operations, and matrix multiplication. Of course, you cannot use \(\backslash\) or \(/\) or any other command that can solve linear equations.

Test your code on simple two- and three dimensional examples.

(c) OPTIONAL: Can you still use the QR factorization of \(A^\top\) together with \(A^\top(AA^\top)^{-1}b\) to compute a minimum norm solution when \(A^\top\) does not have rank \(m\)?