Problem 1: Truncation error (20 points)
Prove that Heun’s method has a local truncation error of order 2:

\[ T_k \equiv T_k(h) = O(h^2). \]

First decompose the error \( hT_k(h) \) in the sum of the errors

\[ S^1_k(h) = x(t_{k+1}) - x(t_k) - \frac{h}{2} \left[ f(t_k, x(t_k)) + f(t_{k+1}, x(t_{k+1})) \right] \]

and

\[ S^2_k(h) = \frac{h}{2} \left[ f(t_{k+1}, x(t_{k+1})) - f(t_{k+1}, x(t_k) + h f(t_k, x(t_k))) \right]. \]

Then apply to both terms arguments seen in class (not necessarily both in the ODE section). State rigorously all assumptions made.

Problem 2: About properties of methods (30 points)
Consider the third order Heun’s method defined by:

\[
\begin{align*}
F^1_k &= f(t_k, x_k), \\
F^2_k &= f(t_k + h/3, x_k + h F^1_k / 3), \\
F^3_k &= f(t_k + 2h/3, x_k + 2h F^2_k / 3), \\
x_{k+1} &= x_k + h \left( \frac{1}{4} F^1_k + \frac{3}{4} F^3_k \right),
\end{align*}
\]

and \( x_0 = x(t_0) \).

(a) Classify this method (explicit/implicit, one step/multistep, Taylor-type/Runge-Kutta). Write it as \( x_{k+1} = x_k + h \Phi(t_k, x_k; h) \), identifying the incremental function \( \Phi \).

(b) Show that when applied to problem \( x'(t) = \lambda x(t), x(0) = x_0 \) this method generates a sequence of the form

\[ x_{k+1} = \left( 1 + h \lambda + \frac{(h \lambda)^2}{2} + \frac{(h \lambda)^3}{6} \right)^{k+1} x_0, \quad k \geq 0. \]

(c) Draw in Matlab the stability region of the method (see how to do it in the Lecture Notes). Is it \( A \)-stable?
Problem 3: About the numerical behavior of methods (20 points)

Write a Matlab code to implement explicit and implicit Euler for the problem $x'(t) = -1000(x(t) - t^2) + 2t$, $0 \leq t \leq 1$, $x(0) = 0$. Solve for $x(1) = 1$. Run the code for $h = 1, 0.1, 0.01, 0.001, 0.00001$. Which is better and why? Does the problem have a unique solution?

Problem 4: About linear multistep methods (15 points)

Consider the following two linear multistep methods:

\[
\begin{align*}
x_{k+2} & = 3x_k - 2x_{k+1} + f(x_k, t_k) + 3f(t_{k+1}, x_{k+1}), \\
x_{k+2} & = \frac{1}{2}[x_k + x_{k+1}] + 2f(t_{k+1}, x_{k+1}).
\end{align*}
\]

For each method, decide if it is (i) zero-stable; (ii) consistent; and if it is consistent, (iii) find its order of accuracy.

Problem 5: About accuracy for Adams methods (15 points)

By counting degrees of freedom, what is the maximum order of accuracy you would expect to achieve with an $m$-step explicit method where $\alpha_m = 1$, $\alpha_{m-1} = -1$? For an implicit method?