Problem 1 (40 points)

In this example you will implement a method to compute the eigenvalues of a matrix using deflation. Given a known eigenvalue of a matrix, a deflation technique produces a matrix of an order less than the original one that contains all eigenvalues of the original one except the one that was eliminated.

(a) Start by implementing in Matlab the Rayleigh quotient method. Your Matlab function should take as input a matrix $A \in \mathbb{C}^{n \times n}$ and output an eigenvalue/eigenvector pair $(x, \lambda)$. Your implementation of this method should be as economical as possible.

(b) Let $x$ be an eigenvector of $A$ associated with an eigenvalue $\lambda$. Let $Q$ be any nonsingular matrix such that $Qx = \mu e_1$, where $\mu$ is a scalar and $e_1$ is the first column of the identity matrix. Why must $\mu$ be $\neq 0$? Show that

$$QAQ^{-1} = \begin{bmatrix} \lambda & h^* \\ 0 & C \end{bmatrix}$$

and that the eigenvalues of $QAQ^{-1}$ are the same as those of $A$ (in other words that this technique is indeed a deflation).

(c) Take now $Q = MP$ where $M$ is an elementary lower triangular matrix (like the L’s used for Gaussian elimination) and $P$ a permutation matrix. Show how to choose $M$ and $P$ such that $MPx$ is a multiple of the first column of the identity. What is the best choice for $P$?

(d) Implement in Matlab a solver capable to compute all eigenvalues of a matrix based on (a-c). Your implementation should be as economical as possible. Try it on the 4 × 4 matrix of the Lecture Notes:

and compare your results with the ones obtained with the QR eigenvalue algorithm.

**Problem 2 (30 points)**

In this problem we will review basic definitions of rates of convergence. Let \[ \{w_k\} \subset \mathbb{C}^n \] be a sequence converging to \[ w^* \]. The sequence converges

- **Sublinearly** when \( \lim_{k \to +\infty} \frac{\|w_{k+1} - w^*\|}{\|w_k - w^*\|} = 1 \).

- **Linearly** when there exists \( r \in (0,1) \) such that \( \frac{\|w_{k+1} - w^*\|}{\|w_k - w^*\|} \leq r \) for all \( k \) (or \( k \) sufficiently large).

- **Superlinearly** when \( \lim_{k \to +\infty} \frac{\|w_{k+1} - w^*\|}{\|w_k - w^*\|} = 0 \).

- **Quadratically** when there exists \( M > 0 \) such that \( \frac{\|w_{k+1} - w^*\|}{\|w_k - w^*\|^2} \leq M \) for all \( k \) (or \( k \) sufficiently large).

(a) In \( \mathbb{R} \) give an example of a sequence that converges to zero quadratically.

(b) In \( \mathbb{R} \) give an example of a sequence that converges to zero superlinearly but not quadratically.

(c) In \( \mathbb{R} \) give an example of a sequence that converges to zero linearly but not superlinearly.

(d) In \( \mathbb{R} \) give an example of a sequence that converges to zero sublinearly but not linearly.

(e) The above rates are the \( q \) rates as opposed to the \( r \) rates, which are introduced now taking the quadratic rate as an example.

The rate of convergence of \( \{w_k\} \subset \mathbb{C}^n \) is said to be \( r \)-quadratic if there exists a sequence \( \{\alpha_k\} \) in \( \mathbb{R} \) converging quadratically to zero such that

\[
\|w_k - w^*\| \leq \alpha_k.
\]

Give an example in \( \mathbb{R} \) of a sequence that converges \( r \)-quadratically but not \( q \)-quadratically.
(f) Proof that if a sequence \( \{y_k\} \subset C^n \) converges q-quadratically to \( y_* \) then all the component sequences \( \{(y_k)_i\} \) converge r-quadratically to \( (y_*)_i \), for all components \( i \in \{1, \ldots, n\} \).

**Problem 3 (30 points)**

In this problem you will implement a solver for root finding and apply it to find the root of \( \arctan(x) = 0 \).

(a) First, for the example \( f = \arctan \), determine for what values of the starting point \( x_0 \) does Newton’s method converge, for what values it diverges, and for what values it cycles without converging. *Hint:* You can apply Newton’s method itself to determine such values! **OPTIONAL:** Can you then give a mathematical argument why do they exist?

(b) Implement a solver that will first carry a few iterations of the bisection method and then switch to Newton’s method.

(c) **OPTIONAL:** Try to see in you can determine a criteria for the switch that will work for all values of \( x_0 \in [-10^5, 10^5] \) not based on the knowledge of the analytical expression of \( f = \arctan \) but only on the values of \( f \) and \( f' \) that are being generated along the iterations.

(d) Observe the (local) quadratic convergence of Newton’s method by plotting \( \|x_{k+1} - x_*\|/\|x_k - x_*\|^2 \) and by plotting \( \|x_k - x_*\| \) in a semi-log scale. Try other examples of your choice (non-quadratic functions) to view this behavior possibly more clearly.

(e) Now, replace Newton’s method by the secant method. First, what is the (local) rate of convergence of secant method? Redo (d) in this case. Do you really observe a worse rate for the secant method?