10-22-2008 CAAM 453

MARK EMBREE'S CAAM 453 MIDTERM REVIEW 10-22-2008

1 Matrix Factorizations

<u>Basic Matrix Facts</u>

- Range, Kernel
- Fundamental Theorem of Linear Algebra
  $Ran(A)(+)Ker(A^*) = C^m, Ran(A) \perp Ker(A^*)$
  $Ran(A*)(+)Ker(A) = C^n, Ran(A^*) \perp Ker(A)$
- Norms - vector norms, matrix norms
  $||A|| = \max_{x \neq 0} \frac{||Ax||}{||x||} = \max_{||x||=1} ||Ax||$
  Unitary invariance of 2-norm:
  - Q is unitary if $Q \in C^{n \times n}$, $Q^*Q = I$
  - $||Qx||_2 = ||x||_2, ||QA||_2 = ||A||_2 = ||AQ||_2$ (does not hold for 1-norm)
- Vector norm axioms
  - positivity: $||x|| \geq 0, ||x|| = 0 \Leftrightarrow x = 0$
  - scaling: $||ax|| = |a|||x||$
  - triangle inequality: $||x + y|| \leq ||x|| + ||y||$
- Submultiplicativity:
  For all induced matrix norms: $||AB|| \leq ||A||||B||$
- Projectors and Reflectors
  - P is a projector $\Leftrightarrow P^2 = P$
  - P is an orthogonal projector $\Leftrightarrow P = P^*, P^2 = P$
  - Householder reflectors: reflects across the $span\{v\}^\perp$
    * $H(v)x = (I - 2P)x$
      $= (I - \frac{vv^*}{v^*v})x$ (Orthog projection onto $span\{v\} : P = \frac{vv^*}{v^*v}$)
    * H(v) is Hermitian, and H(v) is unitary $(H(v)^*H(v) = I)$

- Givens Rotations

$$\begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & cos(\theta) & & & sin(theta) & & & \\ & & & 1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 1 & & & \\ & & -sin(\theta) & & & cos(theta) & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{pmatrix}$$

- Applications of Householder Reflectors and Givens Reflectors
  - QR Factorization

    Write $A \in \Re^{m \times n}$ as:

    $A = QR, Q^*Q = I, Q \in \Re n \times m, R \in \Re^{m \times n}$ upper triangular
  - Householder QR: Reflect 1st column of $A = [a_1 \ldots a_n]$ to $\pm||a_1||e_1$
    Repeat this step on the $(m-1) \times (n-1)$ lower right submatrix of H_1A
  - Givens QR: Applies a Givens rotation to zero out each subdiagonal entry of A.
  - Gram-Schmidt QR -Gram-Schmidt process:

    Given a basis $\{a_1, \ldots, a_n\}, a_j \in C^m$,

    $q_1 = \frac{a_1}{||a_1||_2}$

    $\hat{q}_k = a_k - \sum_{j=1}^{k-1}(q_j q_j^*)a_k$

    $q_k = \frac{\hat{q}_k}{||\hat{q}_k||_2}$

    $\rightarrow [a_1 \ldots a_n] = A = QR = [q_1 \ldots q_n]R, Q \in C^{m \times n}, R \in (n \times n)$.

    Reduced Skinny QR
  - Gram-Schmidt QR = Triangular Orthogonalization,
    Householder, Givens = Orthogonal Triangularization
- Applications of QR
  - Solve $Ax = b, A \in \Re^{n \times n}$ invertible
    $A = QR$
    $\rightarrow QRx = b$

$\rightarrow Rx = (Q^*b)$
Solve via backsubstitution.

Cost of Householder QR: $2mn^2 - 2/3n^3$

Cost of back substitution: n^2 Discrete Least Squares problems: $\min_x ||b - Ax||_2, A \in \Re^{m \times n}, m \geq n$

- Floating Point Arithmetic

  ~~How many bits are in the IEEE Floating point mantissa?~~

  Basic axioms: $fl(X+Y) = (X+Y)(1+\delta), |\delta| \leq \epsilon_{mach}$

  Nothing on the floating point format.

- Stability and conditioning

  Condition number for solving linear systems

$$Ax = b, (A+\delta A)(x+\delta x) = b, \frac{||\delta x||}{||x||} \leq \underbrace{||A||||A^-1||}_{Condition\#} \frac{||\delta A||}{||A||}$$

  log_{10} of condition # is the number of digits you'll lose.
  Could use SVD to construct worst case situation for the condition number.

– HALFWAY POINT –

2 Interpolation

Basic problem:
Given $f \in C[a,b]$, find $p_n \in P_n$
$p_n(x_j) = f(x_j)$
at the prescribed points $x_0, ..., x_n$.
Given a basis $\phi_0, \phi_1, ..., \phi_n$ for P_n.
Write $p_n(x) = \sum_{j=0}^{n} c_j \phi_j(x)$

$p_n(x_k) = \sum_{j=0}^{n} c_j \phi_j(x_k) = f(x_k), k = 0, ..., n$

$$\begin{bmatrix} \phi_0(x_0) & \phi_1(x_0) & \dots & \phi_n(x_0) \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \phi_0(x_n) & \phi_1(x_n) & \dots & \phi_n(x_n) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}$$

Some notable bases for P_n:

- $\phi_j(x) = x^j$ monomial basis $\rightarrow$ Vandermonde matrix $\rightarrow$ ill-conditioned

- $\phi_j(x) = \prod_{l=0}^{j-1}(x - x_l)$ Newton $\rightarrow$ Lower triangular
- $\phi_j(x) = \frac{\prod_{l=0,l\neq j}^{n}(x - x_l)}{\prod_{l=0,l\neq j}^{n}(x_j - x_l)}$ Lagrange, Identity Matrix

Error Formula: If $f \in C^{n+1}[a,b]$, then for each $x \in (a,b)$ there exists some $\xi \in (a,b)$ such that
$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{j=0}^{n}(x - x_j)$
$||f - p_n||_{L^\infty} \leq \frac{||f^{n+1}||_{L^\infty}}{(n+1)!}||\prod_{j=0}^{n}(x - x_j)||_{L^\infty}$
Runge's function: $f(x) = \frac{1}{1+x^2} x \in [-5,5]$ Key Point: High Degree Interpolation at uniformly spaced points can fail to converge.

(Chebyshev points are much better.) Hermite Interpolation (Don't memorize formulas) $h_n \in P_{2_n+1}$ such that:
$h_n(x_j) = f(x_j)$
$h_n'(x_j) = f'(x_j) j = 0, ..., n$ Piecewise Polynomial Approximation

- Piecewise linear (matches $f(x_j) \Rightarrow C^0$)
- Piecewise cubic Hermite (matches $f(x_j), f'(x_j) \Rightarrow C^1$)
- Splines match $f(x_j)$ use extra degrees of freedom to achieve maximum smoothness (Like a razor commercial).

  Basis splines (B-splines) $B_{j,k}(x)$

  "compact support" - zero except for a small subinterval that depends on j and k.
- Trigonometric Interpolation

  Find $t_n \in span\{e^{0x}, e^{ix}, e^{-ix} ... e^{inx}, e^{-inx}\}$

  such that $t_n(x_j) = f(x_j) j = 0, ..., 2n$

  x_j are uniformly spaced on $[0, 2\pi)$

  f is 2π-periodic.

  The coefficients of t_n are the discrete Fourier coefficients of $f(x_0), ..., f(x_{2n})$

3 Approximation Theory

(# SVD problems ≥ 1)

- Discrete Least Squares

  $A \in C^{m\times n}, m \geq n, \min_{x\in C^n} ||b - Ax||_2$

$b = b_R + b_k$
$||b - Ax||_2^2 = ||\underbrace{b_R - Ax}_{\in Ran(A)} + \underbrace{b_k}_{\in Ker(A^*)}||_2^2 = ||b_R - Ax||_2^2 + ||b_k||_2^2$

Whitman - "Do I contradict myself? Very well, then I contradict myself , I am large, I contain multitudes."

Pick $Ax = b_r$
Note that $A^*b = A^*(b_R + b_k) = A^*b_R + \underbrace{A^*b_k}_{=0} = A^*b_R$
Normal Equations: $A^*Ax = A^*b$ $A^*A \in C^{n\times n}$, invertible $\Leftrightarrow$ A is full rank.
Alternative: $A = QR = [Q1Q2]\begin{bmatrix} R_1 \\ 0 \end{bmatrix} = Q_1R_1$
$||b - Ax||_2^2 = ||b - QRx||_2^2 = ||QQ^*b - QRx||_2^2 = ||Q^*b - Rx||_2^2$
$= ||\begin{bmatrix} Q_1^* \\ Q_2^* \end{bmatrix} b - \begin{bmatrix} R_1 \\ 0 \end{bmatrix} x||_2^2 = ||\begin{bmatrix} Q_1^*b - R_1x \\ Q_2^*b \end{bmatrix}||_2^2$
$= ||Q_1^*b - R_1x||_2^2 + ||Q_2^*b||_2^2$
Find x s.t. $R_1x = Q_1^*b$

- Singular Value decomposition $A \in C^{m\times n}$,
  - $A = U\Sigma V^*$
  - $UU^* = I_m$
  - $\Sigma = diag(\sigma_1, ...\sigma_{min(m,n)}$
  - $V^*V = I_n$
  - $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{min(m,n)}$

Dyadic Decomposition: $A = \sum_{j=1}^{r} \sigma_j u_j v_j^*$, r = rank(A)

E.g., if m=n=r=2,
$A = U\Sigma V^* = [U_1U_2]\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}\begin{bmatrix} v_1^* \\ v_2^* \end{bmatrix} = [\sigma_1u_1\sigma_2u_2]\begin{bmatrix} v_1^* \\ v_2^* \end{bmatrix}$
$= \sigma_1u_1v_1^* + \sigma_2u_2v_2^*$

Fundamental subspaces:

$Ran(A) = span\{u_1, ..., u_r\}$
$Ran(A^*) = span\{v_1, ..., v_r\}$
$Ker(A) = span\{u_{r+1}, ..., u_m\}$
$Ker(A^*) = span\{u_{r+1}, ..., u_n\}$ Pseudoinverse: $A^+ = \sum_{j=1}^{r} \frac{1}{\sigma_j} v_j u_j^*$
$= (A^*A)^{-1}A^*$ if A is full rank

Best Low Rank Approximation:

$$\min_{x \in C^{m\times n}, rank(x) \leq k} ||A - X||_2 = \sigma_{k+1}$$

- Continuous Least Squares and Orthogonal Polynomials Problem: $\min\limits_{p \in P_n} ||f - p||_L^2 = \min\limits_{p \in P_n} (\int_a^b (f(x) - p(x))dx)^{1/2}$
  Inner product: $< f, g >= \int_a^b f(x)g(x)w(x)dx$
  Optimal Polynomial: $p_*(x) = \Sigma c_j \phi_j(x)$
  $\rightarrow \begin{bmatrix} <\phi_0, \phi_0> & \dots & <\phi_n, \phi_0> \\ \vdots & \ddots & \vdots \\ <\phi_0, \phi_n> & \dots & <\phi_n, \phi_n> \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} <f, \phi_1> \\ \vdots \\ <f, \phi_n> \end{bmatrix}$
  Best choice for $\phi_0, ...\phi_n$: for orthogonal polynomials
  ϕ_j has exact degree j if $j \neq k$
  $\Rightarrow c_j =< f, \phi_j > / < \phi_j, \phi_j >$
  Can be built via Gram Schmidt.
  Three Term Recurrence!