CAAM 453 · NUMERICAL ANALYSIS I

Examination 1

Posted 20 October 2006. Corrected 23 October 2006.

Due no later than 5pm on Wednesday, 25 October 2006.

Instructions:

1. Time limit: **4 uninterrupted hours**.
2. There are four questions worth a total of 100 points.
   Please do not look at the questions until you begin the exam.
3. You *may not* use any outside resources, such as books, notes, problem sets, friends, calculators, or MATLAB.
4. Please answer the questions thoroughly and justify all your answers.
   Show all your work to maximize partial credit.
5. Print your name on the line below:

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6. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

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7. Staple this page to the front of your exam.

1. [20 points: (a)=5 points; (b)=8 points; (c)=7 points]

   (a) Given a nonzero vector $\mathbf{q} \in \mathbb{C}^n$, write down a formula for the orthogonal projector onto $\operatorname{span}\{\mathbf{q}\}$.

   (b) Write down the (classical) Gram–Schmidt process for generating an orthonormal basis for a set of linearly independent vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$, where $\mathbf{a}_j \in \mathbb{C}^m$ for $j = 1, \ldots, n$ and $m \geq n$.

   (c) Describe, as precisely as possible, how the Gram–Schmidt process can be used to obtain a QR factorization of a matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$. What happens if $\mathbf{A}$ does not have full rank?

2. [25 points: (a)=8 points; (b)=5 points; (c)=5 points; (d)=7 points]

   Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix}.$$

   (a) Compute the singular value decomposition of $\mathbf{A}$.

   (b) Write down a best (2-norm) rank-1 approximation $\mathbf{X}_1$ to $\mathbf{A}$, and specify $\|\mathbf{A} - \mathbf{X}_1\|_2$.

   (c) Consider the rank-1 matrix

$$\mathbf{X} = \sigma_3 \mathbf{u}_1 \mathbf{v}_1^*.$$

   (This is not a typo.) What is $\|\mathbf{A} - \mathbf{X}\|_2$?

   (d) Describe *infinitely many* rank-1 matrices $\mathbf{X}$ for which $\|\mathbf{A} - \mathbf{X}\|_2 = \|\mathbf{A} - \mathbf{X}_1\|_2$.

3. [25 points: (a)=9 points; (b)=8 points; (c)=8 points]

For this problem, suppose $\mathbf{A} \in \mathbb{C}^{m\times n}$, $m \geq n$, has the singular value decomposition

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^* = \sum_{j=1}^{n} \sigma_j \mathbf{u}_j \mathbf{v}_j^*.$$

(a) Suppose that $\mathbf{A}$ has full rank. We have seen that the least squares problem

$$\min_{\mathbf{x}\in\mathbb{C}^n} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2$$

can be solved in several different ways. For example, the optimal $\mathbf{x}$ satisfies the normal equations

$$\mathbf{A}^*\mathbf{A}\mathbf{x} = \mathbf{A}^*\mathbf{b}.$$

Use the normal equations to derive a compact formula for the optimal $\mathbf{x}$ in terms of the singular value decomposition of $\mathbf{A}$.

When the matrix $\mathbf{A} \in \mathbb{C}^{m\times n}$, $m \geq n$ is rank deficient (or nearly so, i.e., $\sigma_n \ll \sigma_1$), it is often appealing to replace the standard least squares problem by the *regularized* problem

$$\min_{\mathbf{x}\in\mathbb{C}^n} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 + \lambda\|\mathbf{x}\|_2^2$$

for some real constant $\lambda > 0$.

(b) Show that the $\mathbf{x}$ that solves this regularized problem satisfies the equation

$$(\mathbf{A}^*\mathbf{A} + \lambda\mathbf{I})\mathbf{x} = \mathbf{A}^*\mathbf{b}.$$

Hint: Apply the normal equations to a larger matrix and vector that contain $\mathbf{A}$ and $\mathbf{b}$.

(c) Use the equation in (b) to derive an expression for the solution $\mathbf{x}$ to the regularized problem in terms of the singular value decomposition of $\mathbf{A}$. (Justify the nonsingularity of any matrices you invert.)

4. [30 points: 6 points for each part]

When we have spoken about norms, projectors, and the like, we have usually been working with vectors and matrices, but these concepts generalize to a much broader setting. In this problem, you will apply these ideas to develop a bound on the accuracy of polynomial interpolation.

Let P_n denote the linear operator that maps $f \in C[a,b]$ to the polynomial p that interpolates f at the distinct points $x_0, \ldots, x_n$, $\{x_j\}_{j=0}^n \subset [a,b]$. In other words, $P_n f = p$, where p is the unique polynomial of degree n (or less) for which $f(x_j) = p(x_j)$ for $j = 0, \ldots, n$.

(a) Explain why P_n is a *projector*.
(What does $P_n p$ equal if p is a polynomial of degree n?)

For the rest of this problem, we use the following norm on $g \in C[a,b]$:

$$\|g\| = \max_{a \le x \le b} |g(x)|.$$

This norm obeys the three familiar norm axioms. It also induces the operator norm

$$\|P_n\| = \sup_{\|f\| \neq 0} \frac{\|P_n f\|}{\|f\|} = \sup_{\|f\|=1} \|P_n f\|.$$

(For purposes of this problem, you can take "sup" to mean the same as "max".)

(b) Show that if $x_0 = a$ and $x_1 = b$, then $\|P_0\| = \|P_1\| = 1$.

(c) Recall that we can write the polynomial $p = P_n f$ in the Lagrange form

$$P_n f = \sum_{j=0}^{n} f(x_j)\ell_j(x).$$

Write down a formula for the functions $\ell_j(x)$.

(d) Prove that $\|P_n\| = \max_{x \in [a,b]} \sum_{j=0}^{n} |\ell_j(x)|$.

(e) Let p_* denote any polynomial of degree n (e.g., the polynomial that minimizes $\|f - p_*\|$).
Prove that $\|f - P_n f\| \le (1 + \|P_n\|)\|f - p_*\|$.
Hint: Add $0 = p_* - p_*$ to the left hand side....