CAAM 453/553 · NUMERICAL ANALYSIS I

Problem Set 1

Posted Wednesday 26 August 2009. Due Friday 4 September 2009.
CAAM 453 students should complete problems 1, 2, 4, 5 [100 points].
CAAM 553 students should complete problems 1, 2, 3, 5, 6, 7. [150 points].
(Students are welcome to attempt more problems if they wish.)

1. [20 points]

   Recall that λ is an eigenvalue of $\mathbf{A}$ if there exists some vector $\mathbf{v} \neq \mathbf{0}$ such that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$.

   Let $\rho(\mathbf{A})$ denote the *spectral radius* of $\mathbf{A}$, i.e.,

   $$\rho(\mathbf{A}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathbf{A}\}.$$

   (a) Show that for any *induced* matrix norm, $\|\mathbf{A}\| \geq \rho(\mathbf{A})$ for all $\mathbf{A} \in \mathbb{C}^{n\times n}$.
   [Trefethen & Bau, exercise 3.2]

   (b) Show that the spectral radius *is not* a matrix norm.
   Which of the three basic norm axioms (positivity, scaling, triangle inequality) fail to hold?

2. [15 points]

   Suppose $\mathbf{D} \in \mathbb{C}^{n\times n}$ is zero everywhere except for the main diagonal, $\mathbf{D} = \mathrm{diag}(d_1, \ldots, d_n)$. Show that $\|\mathbf{D}\|_p = \max_j |d_j|$ for all $p \geq 1$, where $\|\mathbf{D}\|_p$ refers to the matrix norm induced by the vector p-norm

   $$\|\mathbf{x}\|_p = \Big(\sum_{j=1}^{n} |x_j|^p\Big)^{1/p}.$$

3. [20 points] Let $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{A} \in \mathbb{C}^{m\times n}$. Prove the following relationships between the 1- and 2-vector norms, and the matrix norms they induce. In each case, demonstrate a vector or matrix for which equality is satisfied.

   (a) $\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1$.

   (b) $\|\mathbf{x}\|_1 \leq \sqrt{n}\,\|\mathbf{x}\|_2$.

   (c) $\|\mathbf{A}\|_1 \leq \sqrt{m}\,\|\mathbf{A}\|_2$.

   (d) $\|\mathbf{A}\|_2 \leq \sqrt{n}\,\|\mathbf{A}\|_1$.

4. [20 points]

   Consider the vector $\mathbf{x} = (2,\, 3/2)^T$.

   (a) Compute a vector $\mathbf{v}$ such that the Householder reflector $\mathbf{H}(\mathbf{v})$ yields

   $$\mathbf{H}(\mathbf{v})\mathbf{x} = \begin{bmatrix} \|\mathbf{x}\|_2 \\ \mathbf{0} \end{bmatrix}.$$

   (b) Compute $\mathbf{H}(\mathbf{v})^*\mathbf{H}(\mathbf{v})$ for this particular $\mathbf{v}$ to verify that $\mathbf{H}(\mathbf{v})$ is unitary.

   (c) Construct the orthogonal projector $\mathbf{P}$ onto $\mathrm{span}\{\mathbf{v}\}$.

   (d) Produce a precise drawing (or plot) showing $\mathrm{span}\{\mathbf{v}\}$, $\mathrm{span}\{\mathbf{v}\}^\perp$, $\mathbf{x}$, $\mathbf{P}\mathbf{x}$, $(\mathbf{I}-\mathbf{P})\mathbf{x}$, and $\mathbf{H}(\mathbf{v})\mathbf{x}$. Be sure to label your illustration clearly.

continued...

5. [45 points]

For a dense matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, the standard technique for computing the QR decomposition uses Householder reflectors. An alternative approach reduces $\mathbf{A}$ to upper-triangular form using *Givens rotations*. A Givens rotation is a unitary matrix whose entries are zero except as specified:

$$\mathbf{G}(j,k,\theta) = \begin{bmatrix} 1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & 1 & & & & & & & \\ & & & \cos\theta & & & & \sin\theta & & \\ & & & & 1 & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & 1 & & & \\ & & & -\sin\theta & & & & \cos\theta & & \\ & & & & & & & & 1 & \\ & & & & & & & & & \ddots \\ & & & & & & & & & & 1 \end{bmatrix}. \quad \begin{matrix} \leftarrow \textit{row } j \\ \\ \leftarrow \textit{row } k \end{matrix}$$

$$\uparrow \text{ column } j \qquad \uparrow \text{ column } k$$

(a) Prove that the 2×2 Givens rotation

$$\mathbf{G}(1,2,\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

is a unitary matrix for any θ.

(b) Show how to pick θ such that

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \sqrt{\alpha^2+\beta^2} \\ 0 \end{bmatrix}.$$

(c) Design an algorithm for computing the QR factorization $\mathbf{A} = \mathbf{QR}$ using Givens rotations, rather than Householder transformations. In particular, describe the order in which the lower triangular entries of $\mathbf{A}$ (that is, the entries in positions (j,k) for $j > k$) should be eliminated, and those Givens rotations that should be used to achieve this elimination. How many total rotations are necessary? What is the matrix $\mathbf{Q}$?

(d) Write a MATLAB function

```
function [Q,R] = givens_qr(A)
```

that constructs the matrices $\mathbf{Q}$ and $\mathbf{R}$ for $\mathbf{A} \in \mathbb{C}^{m\times n}$ using the algorithm you describe in part (c). Demonstrate that your routine works by reporting `norm(Q*R-A)/norm(A)` for `A = randn(10,5)` and the *final* matrix `R`. *Please do not print out* `Q` *or the values of* `R` *at each step.*

6. [25 points]

Let $|\mathbf{x}|$ denote the entrywise absolute value of a vector $\mathbf{x}$, i.e., if x_j is the jth entry of $\mathbf{x}$, then the jth entry of $|\mathbf{x}|$ is $|x_j|$. We say that $|\mathbf{x}| \le |\mathbf{y}|$ provided $|x_j| \le |y_j|$ for all j.

A vector norm $\|\cdot\|$ is said to be *absolute* provided $\|\mathbf{x}\| = \|\,|\mathbf{x}|\,\|$.

A vector norm $\|\cdot\|$ is said to be *monotone* if $|\mathbf{x}| \le |\mathbf{y}|$ implies $\|\mathbf{x}\| \le \|\mathbf{y}\|$.

Consider the following vector norm on $\mathbb{R}^2$:

$$\|\mathbf{x}\| = |x_1 - x_2| + |x_2|,$$

where $\mathbf{x} = (x_1, x_2)^T$.

(a) Show that this norm satisfies the three vector norm axioms.

(b) Draw the unit ball for this norm (i.e., the set of all $\mathbf{x} \in \mathbb{R}^2$ for which $\|x\| = 1$).

(c) Is this norm absolute?

(d) Is this norm monotone?

(e) Compute the matrix norm induced by this vector norm for the matrix

$$\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

[adapted from Horn and Johnson]

7. [25 points]

Projectors play an essential role in spectral theory, that is, the study of eigenvalues, eigenvectors, and related objects. Suppose $\mathbf{A} \in \mathbb{C}^{n\times n}$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ with associated right eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ and left eigenvectors $\mathbf{y}_1, \ldots, \mathbf{y}_n$ (i.e., $\mathbf{A}\mathbf{x}_j = \lambda_j \mathbf{x}_j$ and $\mathbf{y}_j^* \mathbf{A} = \lambda_j \mathbf{y}_j^*$). Then the *spectral projector* associated with λ_j is defined as

$$\mathbf{P}_j = \frac{\mathbf{x}_j \mathbf{y}_j^*}{\mathbf{y}_j^* \mathbf{x}_j} \in \mathbb{C}^{n\times n}.$$

(a) Compute the spectral projectors for the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for arbitrary constant α. Under what circumstances are these orthogonal projectors?

(b) Verify that $\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 = \mathbf{I}$ and $\mathbf{P}_j \mathbf{P}_k = \mathbf{0}$ when $j \ne k$.
(These properties hold in general for spectral projectors.)

(c) How do the 2-norms of the spectral projectors associated with the eigenvalues $\lambda = -1$ and $\lambda = 1$ relate to the angle between the right eigenvectors associated with these eigenvalues as $\alpha \to \infty$? (A qualitative answer is sufficient; you do not need to explicitly compute the 2-norms.)

(d) [optional] For the matrix $\mathbf{A}$ in part (a), confirm that you obtain the same spectral projectors from the formula

$$\mathbf{P}_j = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_j} (z\mathbf{I} - \mathbf{A})^{-1}\, \mathrm{d}z,$$

where Γ_j is a Jordan curve containing λ_j in its interior and all other eigenvalues in its exterior. (For details on this approach, see Section I.5 of Tosio Kato, *Perturbation Theory for Linear Operators*, corrected 2nd ed., Springer, 1980.)

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Supplemental Problem

This optional extra problem goes beyond the scope of the lectures.

S1. Let $\mathbf{P} \in \mathbb{C}^{n\times n}$ be any (not necessarily orthogonal) projector with $1 \le \operatorname{rank} \mathbf{P} < n$.
Prove that $\|\mathbf{P}\|_2 = \|\mathbf{I} - \mathbf{P}\|_2$.

Hint. This handy theorem has been repeatedly discovered over the years, as described in a very interesting recent survey by Daniel Szyld. Here is an outline of the simplest proof that Szyld provides.

- Explain why one can reduce the problem to proving that $\|\mathbf{Pz}\|_2 \le \|\mathbf{I} - \mathbf{P}\|_2$ for all $\|\mathbf{z}\|_2 = 1$.

To prove this fact, take any $\mathbf{z} \in \mathbb{C}^n$ with $\|\mathbf{z}\|_2 = 1$, and define $\mathbf{x} = \mathbf{Pz}$ and $\mathbf{y} = (\mathbf{I} - \mathbf{P})\mathbf{z}$.

- Explain why $\|\mathbf{Pz}\|_2 \le \|\mathbf{I} - \mathbf{P}\|_2$ if $\mathbf{x} = 0$ or $\mathbf{y} = 0$. (Show that $\|\mathbf{I} - \mathbf{P}\|_2 \ge 1$.)

If $\mathbf{x}$ and $\mathbf{y}$ are both nonzero, then define

$$\widehat{\mathbf{x}} = \frac{\|\mathbf{y}\|_2}{\|\mathbf{x}\|_2}\mathbf{x}, \qquad \widehat{\mathbf{y}} = \frac{\|\mathbf{x}\|_2}{\|\mathbf{y}\|_2}\mathbf{y}, \qquad \mathbf{w} = \widehat{\mathbf{x}} + \widehat{\mathbf{y}}.$$

- Show $\|\mathbf{w}\|_2 = \|\mathbf{z}\|_2$.
- Explain why $\|\mathbf{Pz}\|_2 = \|(\mathbf{I} - \mathbf{P})\mathbf{w}\|_2 \le \|\mathbf{I} - \mathbf{P}\|_2$.