CAAM 453/553 · NUMERICAL ANALYSIS I

Problem Set 6

Posted Friday, 13 November 2009. Due Monday, 23 November 2009.
CAAM 453 students should complete 4 problems (including problem 2).
CAAM 553 students should complete all 5 problems.

1. [25 points: 6 points each for (a), (b), (c); 7 points for (d)]

   Consider the differential equation $x'(t) = \lambda x(t)$ with $x(0) = x_0$.

   (a) Show that when applied to this equation, Heun's method yields

   $$x_{k+1} = (1 + h\lambda + \tfrac{1}{2}h^2\lambda^2)x_n.$$

   (b) Develop an analog of the formula for x_{k+1} in part (a), but now using the classical fourth-order Runge–Kutta method.

   (c) Compare your answers from (a) and (b) to the Taylor series for $x(t_{k+1})$ (that is, the exact solution at t_{k+1}), expanded about the point $t = t_k$.

   (d) Use MATLAB to plot the set of all $h\lambda \in \mathbb{C}$ for which $|x_k| \to 0$ as $k \to \infty$ for Heun's method and the fourth-order Runge–Kutta method.

   [Adapted from Süli and Mayers]

2. [25 points: 5 points for each part]

   Consider 2-step linear multistep methods of the form

   $$x_{k+2} + Ax_{k+1} + Bx_k = hCf_{k+1}$$

   for the initial value problem $x'(t) = f(t, x(t))$, $x(t_0) = x_0$, where A, B, and C are constants.

   (a) Determine all choices of A, B, and C for which this method is consistent.

   (b) Determine a choice of A, B, and C that gives $O(h^2)$ truncation error.

   (c) Assess the zero-stability of the method found in part (b).

   (d) What does your answer to part (c) imply about the behavior of the linear multistep method as $h \to 0$ for such values of A, B, and C?

   (e) For the method found in part (b), calculate those values of λh for which $x_k \to 0$ as $k \to \infty$ when applied to the differential equation $x' = \lambda x$.

Method of Lines. Many physical models give rise to time-dependent partial differential equations. General techniques to solve such problems are beyond the scope of this course. However, many such problems can be attacked using standard ODE integrators via a technique known as the *method of lines.* In the next three problems, you will solve two fundamental equations:

$u_t = u_x$	first-order wave equation
$u_t = u_{xx}$	heat equation.

In these equations, $u \equiv u(t,x)$ is a scalar function of two real variables, u_t denotes the time derivative, and u_x denotes the space derivative.

3. [25 points: 6 points each for (a), (b), (c); 7 points for (d)]

   Consider the first-order wave equation $u_t = u_x$ for $t \geq 0$ and $x \in (-\infty, \infty)$. Given the initial data

$$u(0,x) = \sin(2\pi x),$$

   the exact solution is simply

$$u(t,x) = \sin(2\pi(x+t)).$$

   The method of lines approximates the solution to a partial differential equation by first discretizing the domain in the x direction into points $x_j = j\Delta x$, where $\Delta x = 1/n$ for some fixed n. Since the initial data is periodic, we only need to discretize from $x_1 = \Delta x$ through $x_n = 1$, and then assign $x_0 = x_n$ by periodicity.

   Now approximate the spatial (x) derivative by a finite difference method. The simplest approach is to use the forward difference

$$u_x(t,x_j) \approx \frac{u(t,x_{j+1}) - u(t,x_j)}{\Delta x};$$

   one can use a Taylor series to verify this approximation incurs on $O(\Delta x)$ error.

   The method of lines transforms the partial differential equation $u_t = u_x$ into an ordinary differential equation by replacing u_x with this finite difference approximation:

$$u_t(t,x_j) = \frac{u(t,x_{j+1}) - u(t,x_j)}{\Delta x}.$$

   Exploiting periodicity (which implies that $u(t,x_n) = u(t,x_0)$), we have reduced the partial differential equation to a system of n ordinary differential equations. Using the notation

$$\mathbf{u}(t) = \begin{pmatrix} u(t,x_1) \\ u(t,x_2) \\ \vdots \\ u(t,x_n) \end{pmatrix} \in \mathbb{R}^n,$$

   this system of differential equations can be written as

$$\mathbf{u}_t(t) = \mathbf{A}\mathbf{u}(t).$$

   (a) What is the matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$?
   (Be careful not to neglect the entry that arises because of periodicity.)

   (b) Verify (by proving analytically, or by simply computing a numerical example for some moderate value of n, whichever you prefer) that $\mathbf{A}$ has the n eigenvalues and associated eigenvectors

$$\lambda_j = (\mathrm{e}^{\mathrm{i}\theta_j} - 1)/\Delta x, \qquad \mathbf{v}_j = (\mathrm{e}^{\mathrm{i}\theta_j}, \mathrm{e}^{2\mathrm{i}\theta_j}, \ldots, \mathrm{e}^{n\mathrm{i}\theta_j})^T,$$

   for $\theta_j = 2\pi j/n$ for $j = 1, \ldots, n$.

   Finally, the method of lines solves $\mathbf{u}_t = \mathbf{A}\mathbf{u}$ using an ODE integrator. For simplicity, use the forward Euler method:

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \Delta t \mathbf{A}\mathbf{u}_k.$$

   (c) Consider the eigenvalues from part (b), together with the theory of absolute stability for the forward Euler method, to determine a sharp condition on Δt that ensures there are no exponentially growing solutions for a fixed value of Δx. (You have just derived the famous *CFL condition*, first noted in a seminal 1928 paper by Richard Courant, Kurt Otto Friedrichs, and Hans Lewy.)

(d) Implement your algorithm in MATLAB to confirm that your answer to (c) is correct. In particular, take $\Delta x = 1/50$ ($n = 50$) and give solutions when Δt is (1) twice the maximum and (2) equal to the maximum allowed by the stability requirement from (c).

You may show this data in several ways: You can plot the solution at time $t = 2$ in two dimensions ($u(2, x)$ versus x), or in three dimensions for $t \in [0, 2]$. For the latter, the following MATLAB commands may prove useful: `surf`, `mesh`, `waterfall`, `pcolor`, `shading interp`.

4. [25 points: 6 points each for (a), (b), (c); 7 points for (d)]

It is natural to pursue a better approximation to the exact solution of the first order wave equation by replacing the first-order finite difference approximation to u_x by a more accurate expression, such as the centered difference formula

$$u_x(t, x_j) \approx \frac{u(t, x_{j+1}) - u(t, x_{j-1})}{2\Delta x},$$

which one can verify is $O((\Delta x)^2)$ accurate.

(a) As in Problem 2, determine the matrix $\mathbf{A}$ that arises when this centered-difference formula replaces u_x in $u_t = u_x$ to give rise to the system of ODEs $\mathbf{u}_t = \mathbf{A}\mathbf{u}$.

(b) Where in the complex plane do the eigenvalues of $\mathbf{A}$ fall?
(You can simply include a plot for one example of moderate size, or you can prove your claim. Hint: your matrix should be skew-symmetric, $\mathbf{A} = -\mathbf{A}^*$.)

(c) Again, we will solve $\mathbf{u}_t = \mathbf{A}\mathbf{u}$ using the forward Euler method. Use your answer from (b) to explain whether you expect this second order approach to be superior to the first order method from Question 2. In particular, give a suitable choice of Δt that will prevent exponentially growing parasitic solutions, or state that no such Δt exists.

(d) Present computational evidence from MATLAB to confirm your answer to part (c).

5. [25 points: 6 points each for (a), (b), (c); 7 points for (d)]

Now apply the method of lines to the heat equation, $u_t = u_{xx}$ for $t \geq 0$ and $x \in [0, 1]$ with homogeneous boundary conditions, $u(t, 0) = u(t, 1) = 0$. For the spatial derivative, use the second order approximation to the second derivative,

$$u_{xx}(t, x_j) \approx \frac{u(t, x_{j-1}) - 2u(t, x_j) + u(t, x_{j+1})}{(\Delta x)^2}.$$

(a) As usual, reduce the problem to the form $\mathbf{u}_t = \mathbf{A}\mathbf{u}$. What is the matrix $\mathbf{A}$?
Note: Exploiting the homogeneous boundary conditions $u(t, 0) = u(t, 1) = 0$, one can define

$$\mathbf{u}(t) = (u(t, x_1),\ u(t, x_2),\ \ldots,\ u(t, x_{n-1}))^T \in \mathbb{R}^{n-1}$$

so that $\mathbf{A} \in \mathbb{R}^{(n-1)\times(n-1)}$.

(b) Verify (numerically or analytically, as you prefer) that $\mathbf{A} \in \mathbb{R}^{(n-1)\times(n-1)}$ has the eigenvalues

$$\lambda_j = \frac{2\cos(\theta_j) - 2}{(\Delta x)^2}$$

for $j = 1, \ldots, n-1$, where $\theta_j = j\pi/n$.

(c) Again, we will solve $\mathbf{u}_t = \mathbf{A}\mathbf{u}$ using the explicit Euler method. What is required of Δt to ensure there are no exponentially growing parasitic modes?

(d) Implement this method in MATLAB to confirm your value for Δt when $\Delta x = 1/20$. Integrate over $t \in [0, 0.1]$ and $x \in [0, 1]$ with the initial condition

$$u_0(x) = \begin{cases} 1, & x \in [\frac{2}{5}, \frac{3}{5}]; \\ 0, & \text{otherwise,} \end{cases}$$

and boundary conditions

$$u(t, 0) = u(t, 1) = 0.$$

Show plots for both Δt just within the stability requirement, and just barely beyond it.

Supplemental Problem

S1. Suppose Euler's method is used to solve the initial value problem

$$x'(t) = f(x, t), \qquad x(t_0) = x_0$$

on the interval $t \in [t_0, t_{\text{final}}]$. Your goal is to compute $x(t_{\text{final}})$ as accurately as possible.

(a) Describe how Richardson extrapolation can be applied to this problem. Write a MATLAB code to implement the procedure. You may use the code `euler.m` from the class web site.

(b) Apply your code to $x'(t) = x(t)$ with $x(0) = 1$ on the interval $[0, 1]$. Start with step size $h = 1$, and state the number of levels your extrapolation table requires to give $x(1)$ to within 10^{-10} of the exact solution.

(c) You are designing a space probe to rendezvous with a near-earth asteroid. The encounter is to take place at time $t = 1$, and you require an accurate position vector for the asteroid at that time, so you know where to send your probe for the rendezvous. Suppose the motion of this asteroid, in suitably normalized coordinates, is given by

$$\mathbf{x}''(t) = -\frac{\mathbf{x}(t)}{\|\mathbf{x}(t)\|^3}, \qquad \mathbf{x}(0) = \begin{bmatrix} .2 \\ 0 \end{bmatrix}, \quad \mathbf{x}'(0) = \begin{bmatrix} -.65 \\ 2.5 \end{bmatrix},$$

where $\mathbf{x}(t) \in \mathbb{R}^2$ denotes the position of the satellite at time t. Write this second order system as a first order system, integrate this orbit with Euler's method, and apply your extrapolation technique to obtain the components of $\mathbf{x}(1)$ to decent precision. Start your table with time step $h = 2^{-5}$.

(d) Compare this extrapolated value to the answer at t_{final} that you obtain using the Störmer–Verlet method for this conservative system. For the equation $\mathbf{x}''(t) = \mathbf{f}(\mathbf{x}(t))$ with $\mathbf{x}_0 = \mathbf{x}(t_0)$ and $\mathbf{v}_0 = \mathbf{x}'(t_0)$, this method is:

$$\begin{aligned} \mathbf{x}_{k+1/2} &= \mathbf{x}_k + \tfrac{1}{2}h\mathbf{v}_k \\ \mathbf{v}_{k+1} &= \mathbf{v}_k + h\mathbf{f}(\mathbf{x}_{k+1/2}) \\ \mathbf{x}_{k+1} &= \mathbf{x}_{k+1/2} + \tfrac{1}{2}h\mathbf{v}_{k+1}. \end{aligned}$$