CAAM 453 FALL 2006 EXAM 1 KEY

1 a.
$$\frac{qq^*}{q^*q}$$

b. Gram Schmidt

$$q_1 = a_1 / \|a_1\|_2$$

For $k = 2 : n$

$$\hat{q}_k = a_k - \sum_{j=0}^{k-1} \frac{q_j q_j^*}{q_j^* q_j} a_k$$

$$q_k = \hat{q}_k / \|\hat{q}_k\|$$

c. Rearrange (b) to obtain

$$a_1 = q_1 \|a_1\|_2$$

$$a_2 = q_2 \|\hat{q}_2\|_2 + q_1 \left(\frac{q_1^* a_2}{q_1^* q_1} \right)$$

$$\vdots$$

$$a_n = q_n \|\hat{q}_n\|_2 + \sum_{j=1}^{n-1} q_j \left(\frac{q_j^* a_n}{q_j^* q_j} \right)$$

In matrix form

$$\begin{pmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \|a_1\|_2 & \frac{q_1^* a_2}{q_1^* q_1} & \cdots & \frac{q_1^* a_n}{q_1^* q_1} \\ 0 & \|\hat{q}_2\|_2 & & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots \; 0 & \|\hat{q}_n\|_2 \end{pmatrix}$$

$$A = Q \; R$$

$$A \in \mathbb{C}^{m \times n}, \quad Q \in \mathbb{C}^{m \times n}, \quad R \in \mathbb{C}^{n \times n}$$

If A does not have full rank, then eventually $\hat{q}_j = 0$ for some j (or $a_1 = 0$), giving a zero on the diagonal of R.

2 a. $A^*A = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$

$\Rightarrow \quad \sigma_1 = \sqrt{8} = 2\sqrt{2}$

$\sigma_2 = \sqrt{4} = 2$

$\sigma_3 = \sqrt{2}$

By inspection, eigenvectors of A^*A are

$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow \quad u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{8}} \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$u_3 = \frac{1}{\sigma_3} A v_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$A = U \Sigma V^* = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2\sqrt{2} & & \\ & 2 & \\ & & \sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^*$

b. $X_1 = \sigma_1 u_1 v_1^* = \sqrt{8} \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right) (0 \; 0 \; 1) = 2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

$\|A - X_1\| = \sigma_2 = 2$

c. $X = \sigma_3 u_1 v_1^* \Rightarrow \quad A - X = \sum_{j=1}^{3} \sigma_j u_j v_j^* - \sigma_3 u_1 v_1^*$

$= \underbrace{(\sigma_1 - \sigma_3)}_{=\sqrt{2}} u_1 v_1^* + \underbrace{\sigma_2}_{2} u_2 v_2^* + \underbrace{\sigma_3}_{\sqrt{2}} u_3 v_3^*$

$\Rightarrow \|A - X\| = 2$

d. if $X = \sigma u_1 v_1^*$ with $|\sigma_1 - \sigma| \le 2$, i.e.,

$\sigma_1 - 2 \le \sigma \le \sigma_1 + 2$

then $\|A - X\| = \sigma_2 = 2$.

3a $A \in \mathbb{C}^{m \times n}$, $m \geq n$, A full rank.

Normal equations: $A^*Ax = A^*b \Rightarrow x = (A^*A)^{-1}A^*b$.

$A = \hat{U}\hat{\Sigma}V^*$ (economy-sized SVD)

$A^*A = (V\hat{\Sigma}^*\hat{U}^*)(\hat{U}\hat{\Sigma}V) = V\hat{\Sigma}^2V^*$; $(A^*A)^{-1} = V\hat{\Sigma}^{-2}V^*$

$(A^*A)^{-1}A^* = V\hat{\Sigma}^{-2}V^*V\hat{\Sigma}^*\hat{U}^* = V\hat{\Sigma}^{-1}\hat{U}^*$

$$\boxed{x = V\hat{\Sigma}^{-1}U^*b = \sum_{j=1}^{n} \frac{1}{\sigma_j} v_j u_j^* b}$$

3b Notice that

$$\begin{pmatrix} A \\ \sqrt{\lambda} I \end{pmatrix} x - \begin{pmatrix} b \\ 0 \end{pmatrix} = \begin{pmatrix} Ax \\ \sqrt{\lambda}x \end{pmatrix} - \begin{pmatrix} b \\ 0 \end{pmatrix}$$

$$\Rightarrow \left\| \begin{pmatrix} A \\ \sqrt{\lambda} I \end{pmatrix} x - \begin{pmatrix} b \\ 0 \end{pmatrix} \right\|_2^2 = \left\| \begin{pmatrix} Ax - b \\ \sqrt{\lambda}x \end{pmatrix} \right\|_2^2 = \|Ax-b\|_2^2 + \lambda\|x\|_2^2$$

$$\begin{pmatrix} A \\ \sqrt{\lambda} I \end{pmatrix}^* \begin{pmatrix} A \\ \sqrt{\lambda} I \end{pmatrix} x = \begin{pmatrix} A \\ \sqrt{\lambda} I \end{pmatrix}^* \begin{pmatrix} b \\ 0 \end{pmatrix}$$

$$\Rightarrow (A^*A + \lambda I)x = A^*b$$

3c. $A^*A + \lambda I = V(\hat{\Sigma}^{*2} + \lambda I)V^*$

Note: All entries on the diagonal of $\hat{\Sigma}^2 + \lambda I$ are <u>positive</u> even if some corresponding entries of $\hat{\Sigma}^2$ are zero $\Rightarrow (\hat{\Sigma}^2 + \lambda I)^{-1}$ exists.

$$\begin{aligned} \Rightarrow \quad x &= (A^*A + \lambda I)^{-1}A^*b \\ &= V(\hat{\Sigma}^2 + \lambda I)^{-1}V^*V\hat{\Sigma}\hat{U}^*b \\ &= V(\hat{\Sigma}^2 + \lambda I)^{-1}\hat{\Sigma}\hat{U}^*b \\ &= \sum_{j=1}^{n} \frac{\sigma_j}{\sigma_j^2 + \lambda} v_j u_j^* \end{aligned}$$

4 a. P_n is a projector because $P_n^2 = P_n$.

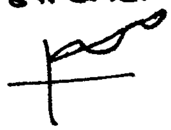
That is, for all $f \in C[a,b]$,

$$P_n^2 f = P_n(P_n f) = P_n p = p = P_n f.$$

b. $\underline{n=0.}$ $P_0 f = f(x_0) = f(a)$

$$\|P_0\| = \sup_{\|f\|\neq 0} \frac{\|P_0 f\|}{\|f\|} = \frac{|f(a)|}{\max_{x\in[a,b]} |f(x)|} \leq 1$$

Equality is attained when $f(x) = f(a) \neq 0$. (constant)



$\underline{n=1}$

$$\|P_1\| = \sup_{\|f\|\neq 0} \frac{\|P_1 f\|}{\|f\|} = \frac{\max\{|f(a)|, |f(b)|\}}{\max_{x\in[a,b]} |f(x)|} \leq 1$$

Again, equality is attained for $f(x) = \text{constant} \neq 0$

c. $$\ell_j(x) = \prod_{\substack{k=0\\k\neq j}}^{n} \frac{x - x_k}{x_j - x_k}.$$

d. $$\|P_n\| = \sup_{\|f\|\neq 0} \frac{\|P_n f\|}{\|f\|} = \sup_{\|f\|\neq 0} \frac{\left\|\sum_{j=0}^{n} f(x_j)\,\ell_j(x)\right\|}{\|f\|}$$

$$\leq \sup_{\|f\|\neq 0} \frac{\left\|\sum_{j=0}^{n} |f(x_j)|\,|\ell_j(x)|\right\|}{\|f\|}$$

$$\leq \sup_{\|f\|\neq 0} \frac{\left(\max_{x\in[a,b]} \|f\|\right) \left\|\sum_{j=0}^{n} |\ell_j(x)|\right\|}{\|f\|} = \left\|\sum_{j=0}^{n} |\ell_j(x)|\right\|$$

Equality is attained if $f = \text{constant} \neq 0$.

e. $$\|f - p_n\| = \|f - P_n f\|$$

$$= \|f - p_* + p_* - P_n f\|$$

$$\leq \|f - p_*\| + \|P_n f - p_*\|$$

$$= \|f - p_*\| + \|P_n f - P_n p_*\| \quad [\text{since } p_* \in \mathcal{P}_n,\ P_n p_* = p_*.]$$

$$\leq \|f - p_*\| + \|P_n\|\,\|f - p_*\|$$

$$= (1 + \|P_n\|)\,\|f - p_*\|.$$