CAAM 453/553 • EXAM 1 • FALL 2009

1a $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

Compute SVD of A^*: $AA^* = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

$\det(\lambda I - AA^*) = (\lambda-5)^2 - 16 = \lambda^2 - 10\lambda + 9 = (\lambda-9)(\lambda-1)$

Eigenvalues λ_j, vectors u_j of AA^*: $\sigma_1 = \sqrt{\lambda_1} = \sqrt{9} = 3$, $u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\sigma_2 = \sqrt{\lambda_2} = \sqrt{1} = 1$, $u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Then $v_1 = \dfrac{A^* u_1}{\sigma_1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $v_2 = \dfrac{A^* u_2}{\sigma_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Define $v_3 \perp \operatorname{span}\{v_1, v_2\}$: $v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

$$A = \begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix}\begin{bmatrix} v_1^* \\ v_2^* \\ v_3^* \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

1b $A = U\Sigma V^* = \sum_{k=1}^{m} \sigma_k u_k v_k^*$

$$AA^* = \Big(\sum_{k=1}^{m} \sigma_k u_k v_k^*\Big)\Big(\sum_{j=1}^{m} \sigma_j u_j v_j^*\Big) = \sum_{j,k=1}^{m} \sigma_j \sigma_k u_j v_j^* u_k v_k^* = \sum_{k=1}^{m} \sigma_k^2 u_k v_k^*.$$

$$(AA^*)^{-1} = \sum_{k=1}^{m} \frac{1}{\sigma_k^2} v_k u_k^* \qquad A^*(AA^*)^{-1} = \Big(\sum_{j=1}^{m} \sigma_j v_j u_j^*\Big)\Big(\sum_{k=1}^{m} \frac{1}{\sigma_k^2} u_k v_k^*\Big)$$

$$= \sum_{j,k=1}^{m} \frac{\sigma_j}{\sigma_k^2} v_j u_j^* u_k v_k^* = \sum_{k=1}^{m} \frac{1}{\sigma_k} v_k u_k^*.$$

1c $\operatorname{Ran}(A) = \operatorname{span}\{u_1, \dots, u_m\}$, $\operatorname{Ker}(A^*) = \{0\}$ since $\operatorname{rank}(A) = m$.

$\operatorname{Ran}(A^*) = \operatorname{span}\{v_1, \dots, v_m\}$, $\operatorname{Ker}(A) = \operatorname{span}\{v_{m+1}, \dots, v_n\}$

1d $Ax = b$. Write $x \in \mathbb{C}^n$ as $x = \sum_{k=1}^{n} c_k v_k$ (since $\operatorname{span}\{v_1, \dots, v_n\} = \mathbb{C}^n$).

$$Ax = b \Rightarrow \Big(\sum_{j=1}^{m} \sigma_j u_j v_j^*\Big)\Big(\sum_{k=1}^{n} c_k v_k\Big) = \sum_{j=1}^{m}\sum_{k=1}^{n} \sigma_j c_k u_j v_j^* v_k = \sum_{j=1}^{m} \sigma_j c_j u_j.$$

Take inner product with u_k to see:

$$u_k^* A x = u_k^* b \Rightarrow \sigma_k c_k u_k^* u_k = u_k^* b \Rightarrow \boxed{c_k = \frac{u_k^* b}{\sigma_k},\ k = 1, \dots, m}$$

$c_{m+1}, \dots, c_n$ can be <u>arbitrary</u>: $x = \sum_{k=1}^{m} \dfrac{u_k^* b}{\sigma_k} v_k + \sum_{k=m+1}^{n} c_k v_k$.

Smallest norm: $c_{m+1} = \cdots = c_n = 0$: $x = \sum_{k=1}^{m} \dfrac{u_k^* b}{\sigma_k} v_k$.

1e $x = \frac{1}{\sigma_1}(u_1^* b) v_1 + \frac{1}{\sigma_2}(u_2^* b) v_2 + c_3 v_3 = \frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$: Smallest norm when $c_3 = 0$.

2a

$$H = I - \frac{2vv^*}{v^*v}$$

2b

If $Hx = \lambda x$, then $\lambda x = \left(I - \frac{2vv^*}{v^*v}\right)x = x - 2v\frac{v^*x}{v^*v} = x - \left(\frac{2v^*x}{v^*v}\right)v$.

Hence, $x - \left(\frac{2v^*x}{v^*v}\right)v \in \operatorname{span}\{x\} \Rightarrow \left(2\frac{v^*x}{v^*v}\right)v \in \operatorname{span}\{x\}$

This is only possible if: $\begin{cases} v = x \Rightarrow \lambda x = x - \frac{2x^*x}{x^*x}x = -x \Rightarrow \boxed{\lambda = -1} \\ \text{-or-} \\ v \perp x \Rightarrow \lambda x = x - 0v = x \Rightarrow \boxed{\lambda = 1} \end{cases}$

Hence: eigenvalue/eigvector pairs: $\lambda = -1,\ x = v$ } 1 eigenpair
$\lambda = 1,\ x \perp v$ } $n-1$ eigenpairs

Geometrically, $\lambda = -1$, $x = v$: H reflects v across $\operatorname{span}\{v\}^\perp$, so the eigenvector v flips directions, from v to $Hv = -v$.

2c

$$Gx = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda x.$$

Characteristic polynomial for G: $\det(\lambda I - G) = (\lambda - \cos\theta)^2 + \sin^2\theta$
$= \lambda^2 - 2\lambda\cos\theta + (\cos^2\theta + \sin^2\theta)$
$= \lambda^2 - 2\lambda\cos\theta + 1$

Roots of $\lambda^2 - 2\lambda\cos\theta + 1$:

$$\frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \frac{2\cos\theta \pm 2\sqrt{-\sin^2\theta}}{2}$$

$$= \cos\theta \pm i\sin\theta = e^{\pm i\theta}.$$

Eigenvectors: $(G - e^{\pm i\theta}I)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\theta - e^{\pm i\theta} & \sin\theta \\ -\sin\theta & \cos\theta - e^{\pm i\theta} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \mp i\sin\theta & \sin\theta \\ -\sin\theta & \mp i\sin\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\Rightarrow \sin\theta\begin{bmatrix} \mp i & 1 \\ -1 & \mp i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$ $\sin\theta = 0 \Rightarrow \theta = 0, \pi, 2\pi, \ldots$
$\Rightarrow$ Double eigenvalue at $+1$ or -1
$\Rightarrow$ eigenspace $= \mathbb{C}^2$.

or

$\mp i x_1 + x_2 = 0 \Rightarrow x_2 = \pm i x_1$

$\Rightarrow \lambda_1 = e^{i\theta}$, $x = \begin{bmatrix} 1 \\ i \end{bmatrix}$
$\lambda_2 = e^{-i\theta}$, $x = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

2d

$$G_{1,2} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad G_{1,3} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$G_{2,3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{bmatrix}$$

2e

$R = G_{1,2}G_{1,3}G_{2,3}$ is the product of unitary matrices, so it must be unitary:

$$R^*R = G_{2,3}^*\,G_{1,3}^*\,G_{1,2}^*\,G_{1,2}\,G_{1,3}\,G_{2,3} = I$$

$Rx = 1x \Rightarrow x$ is not rotated by $R \Rightarrow x$ is the <u>axis of rotation</u>.

3a $B_{j,k}(x) \begin{cases} \neq 0 & \text{when } x \in (x_j, x_{j+k+1}) \\ = 0 & \text{when } x \notin (x_j, x_{j+k+1}) \end{cases}$ $(k>1)$ | $B_{j,0}(x) \begin{cases} \neq 0 & \text{when } x \in [x_j, x_{j+1}) \\ = 0 & \text{when } x \notin [x_j, x_{j+1}) \end{cases}$

3b $B_{j,k} \in C^p(\mathbb{R})$ for $p = k-1$.

3c $B_{j,3}$ is supported on 4 subintervals.

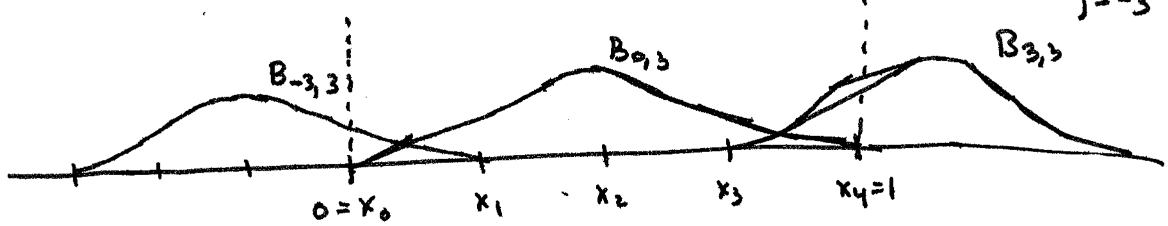
$B_{-3,3}(x)$ is supported on $[x_0, x_1)$

$B_{-2,3}(x)$ is supported on $[x_0, x_2)$

$\vdots$

$B_{3,3}(x)$ is supported on $(x_3, x_4]$

$$S(x) = \sum_{j=-3}^{3} c_j B_{j,3}(x) = \sum_{j=-3}^{n-1} c_j B_{j,3}(x)$$



3d We want to find $c_{-3}, \dots, c_3$ such that

$$S(\xi_j) \approx f(\xi_j) \qquad j = 0, \dots, 2n = 8$$

$$\begin{bmatrix} B_{-3,3}(\xi_0) & \cdots & B_{3,3}(\xi_0) \\ B_{-3,3}(\xi_1) & \cdots & B_{3,3}(\xi_1) \\ \vdots & \ddots & \vdots \\ B_{-3,3}(\xi_8) & \cdots & B_{3,3}(\xi_8) \end{bmatrix} \begin{bmatrix} c_{-3} \\ c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} \approx \begin{bmatrix} f(\xi_0) \\ f(\xi_1) \\ \vdots \\ f(\xi_8) \end{bmatrix}$$

$A \in \mathbb{R}^{(2n+1)\times(n+3)}$ $\qquad c \in \mathbb{R}^{n+3}$ $\qquad f \in \mathbb{R}^{2n+1}$

$A \in \mathbb{R}^{9\times 7}$ $\qquad c \in \mathbb{R}^{7}$ $\qquad f \in \mathbb{R}^{9}$

3e

$$A = \begin{bmatrix} x & x & x & 0 & 0 & 0 & 0 \\ x & x & x & x & 0 & 0 & 0 \\ 0 & x & x & x & 0 & 0 & 0 \\ 0 & x & x & x & x & 0 & 0 \\ 0 & 0 & x & x & x & 0 & 0 \\ 0 & 0 & x & x & x & x & 0 \\ 0 & 0 & 0 & x & x & x & 0 \\ 0 & 0 & 0 & x & x & x & x \\ 0 & 0 & 0 & 0 & x & x & x \end{bmatrix}$$

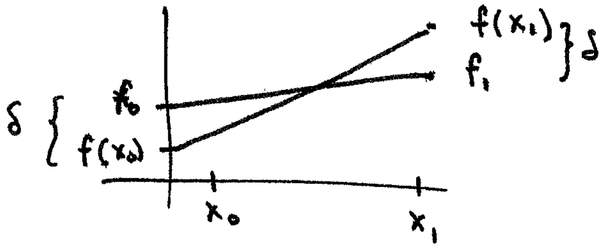
3 or 4 nonzeros per row.

Better to compute the QR factorization of A via Givens rotations as n gets bigger, since most subdiagonal entries are already zero.

4a

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j) \quad \text{for some } \xi \in [a,b]$$

4b



$$\max_{x \in [x_0, x_1]} |p(x) - \hat{p}(x)| \le \delta$$

4c

$$\max_{x \in [x_0, x_1]} |f(x) - \hat{p}(x)| = \max_{x \in [x_0, x_1]} |f(x) - p(x) + p(x) - \hat{p}(x)|$$

$$\le \max_{x \in [x_0, x_1]} |f(x) - p(x)| + \max_{x \in [x_0, x_1]} |p(x) - \hat{p}(x)|$$

$$\le \left(\max_{\xi \in [x_0, x_1]} \frac{|f^{(n+1)}(\xi)|}{(n+1)!} \right) |x_0 - x_1|^2 + \delta$$

$(n=2)$

$$= \max_{\xi \in [x_0, x_1]} \frac{|f''(\xi)|}{2} |x_0 - x_1|^2 + \delta$$

4d

r interpolates an <u>approximation</u> to $f(\tilde{x}, y_0)$ and $f(\tilde{x}, y_1)$. Suppose that approximation is accurate to within δ. Then

$$|r(\tilde{y}) - f(\tilde{x}, \tilde{y})| \le \max_{\eta \in [x_0, x_1]} \frac{|f_{yy}(\tilde{x}, \eta)|}{2} |y_0 - y_1|^2 + \delta$$

What is δ? This term reflects the accuracy of approximation in $p(x)$ and $q(x)$:

$$|p(\tilde{x}) - f(\tilde{x}, y_0)| \le \max_{\xi \in [x_0, x_1]} \frac{|f_{xx}(\xi, y_0)|}{2} |x_0 - x_1|^2$$

$$|q(\tilde{x}) - f(\tilde{x}, y_1)| \le \max_{\xi \in [x_0, x_1]} \frac{|f_{xx}(\xi, y_1)|}{2} |x_0 - x_1|^2$$

Define
$$\delta := \max_{\substack{\xi \in [x_0, x_1] \\ \eta \in [y_0, y_1]}} \frac{|f_{xx}(\xi, \eta)|}{2} |x_0 - x_1|^2$$

Hence

$$|r(\tilde{y}) - f(\tilde{x}, \tilde{y})| \le \max_{\substack{\xi \in [x_0, x_1] \\ \eta \in [y_0, y_1]}} \frac{|f_{yy}(\xi, \eta)|}{2} |y_0 - y_1|^2 + \delta.$$

5a

$$Pf = \frac{\langle f, \phi \rangle}{\langle \phi, \phi \rangle} \phi \Rightarrow P(Pf) = \frac{\left\langle \frac{\langle f, \phi \rangle}{\langle \phi, \phi \rangle} \phi, \phi \right\rangle}{\langle \phi, \phi \rangle} \phi \quad \left[\frac{\langle f, \phi \rangle}{\langle \phi, \phi \rangle} \text{ is a scalar} \right]$$

$$= \frac{\langle f, \phi \rangle}{\langle \phi, \phi \rangle} \frac{\langle \phi, \phi \rangle}{\langle \phi, \phi \rangle} \phi = \frac{\langle f, \phi \rangle}{\langle \phi, \phi \rangle} \phi = Pf$$

Since $\frac{\langle f, \phi \rangle}{\langle \phi, \phi \rangle}$ is a scalar, $\frac{\langle f, \phi \rangle}{\langle \phi, \phi \rangle} \phi \in \operatorname{span}\{\phi\}$.

5b

Gram-Schmidt

$$\phi_0 = P_0$$

$$\phi_j = P_j - \sum_{k=0}^{j-1} \frac{\langle P_j, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle} \phi_k$$

(Subtract orthogonal projections of P_j onto $\phi_0, \ldots, \phi_{j-1}$)

5c

$$\langle e^{ikx}, e^{ilx} \rangle = \int_0^{2\pi} e^{ikx} e^{-ilx} \, dx \qquad (k \neq l)$$

$$= \int_0^{2\pi} e^{i(k-l)x} \, dx = \left[\frac{e^{i(k-l)x}}{k-l} \right]_0^{2\pi}$$

$$= \frac{e^{i(k-l)2\pi} - 1}{k-l} = \frac{1-1}{k-l} = 0.$$

$e^{i(k-l)2\pi} = 1$ since $(k-l)2\pi$ is an integer multiple of 2π.

5d

$$t_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$$

$$0 = \langle f - t_n, e^{ilx} \rangle \Rightarrow \langle t_n, e^{ilx} \rangle = \langle f, e^{ilx} \rangle \quad (*)$$

But $\langle t_n, e^{ilx} \rangle = \left\langle \sum_{k=-n}^{n} c_k e^{ikx}, e^{ilx} \right\rangle = \sum_{k=-n}^{n} c_k \langle e^{ikx}, e^{ilx} \rangle$

$$= c_l \langle e^{ilx}, e^{ilx} \rangle$$

$$(*) \Rightarrow c_l \langle e^{ilx}, e^{ilx} \rangle = \langle f, e^{ilx} \rangle$$

$$\Rightarrow c_l = \frac{\langle f, e^{ilx} \rangle}{\langle e^{ilx}, e^{ilx} \rangle} = \frac{\langle f, e^{ilx} \rangle}{2\pi}$$

since $\langle e^{ilx}, e^{ilx} \rangle = \int_0^{2\pi} e^{i(l-l)x} \, dx = \int_0^{2\pi} 1 \, dx = 2\pi$.