Lecture 1: Introduction to Numerical Analysis

We model our world with continuous mathematics. Whether our interest is natural science, engineering, even finance and economics, the models we most often employ are functions of real variables. The equations can be linear or nonlinear, involve derivatives, integrals, combinations of these and beyond. The tricks and techniques one learns in algebra and calculus for solving such systems exactly cannot tackle the complexities that arise in serious applications. Exact solution may require an intractable amount of work; worse, for many problems, it is impossible to write down an exact solution using elementary functions like polynomials, roots, trig functions, and logarithms.

This course tells a marvelous success story. Through the use of clever algorithms, careful analysis, and speedy computers, we are able to construct *approximate* solutions to these otherwise intractable problems with remarkable speed. Trefethen defines *numerical analysis* to be 'the study of algorithms for the problems of continuous mathematics'.[†] This course takes a tour through many such algorithms, sampling a variety of techniques suitable across many applications. We aim to assess alternative methods based on both accuracy and efficiency, to discern well-posed problems from ill-posed ones, and to see these methods in action through computer implementation.

Perhaps the importance of numerical analysis can be best appreciated by realizing the impact its disappearance would have on our world. The space program would evaporate; aircraft design would be hobbled; weather forecasting would again become the stuff of soothsaying and almanacs. The ultrasound technology that uncovers cancer and illuminates the womb would vanish. Google couldn't rank web pages. Even the letters you are reading, whose shapes are specified by polynomial curves, would suffer. (Several important exceptions involve discrete, not continuous, mathematics: combinatorial optimization, cryptography and gene sequencing.)

On one hand, we are interested in *complexity*: we want algorithms that minimize the number of calculations required to compute a solution. But we are also interested in the *quality* of our approximation: since we do not obtain exact solutions, we must understand the accuracy of our answers. Discrepancies arise from approximating a complicated function by a polynomial, a continuum by a discrete grid of points, or the real numbers by a finite set of floating point numbers. Different algorithms for the same problem will differ in the quality of their answers and the labor required to obtain those answers; we will learn how to evaluate algorithms according to these criteria.

Numerical analysis forms the heart of 'scientific computing' or 'computational science and engineering,' fields that also encompass the high-performance computing technology that makes our algorithms practical for problems with millions of variables, visualization techniques that illuminate the data sets that emerge from these computations, and the applications that motivate them.

Though numerical analysis has flourished in the past sixty years, its roots go back centuries, where numerical approximations were necessary for foundational work in celestial mechanics and, more generally, 'natural philosophy'. Science, commerce, and warfare magnified the need for numerical analysis, so much so that the early twentieth century spawned the profession of 'computers,' people (often women) who conducted computations with hand-crank desk calculators. But numerical analysis has always been more than mere number-crunching, as observed by Alston Householder in the introduction to his *Principles of Numerical Analysis*, published in 1953, the end of the human computer era:

[†] We highly recommend L. N. Trefethen's essay, 'The Definition of Numerical Analysis', (reprinted on pages 321–327 of Trefethen & Bau, *Numerical Linear Algebra*), which inspires our present manifesto.

> The material was assembled with high-speed digital computation always in mind, though many techniques appropriate only to "hand" computation are discussed.... How otherwise the continued use of these machines will transform the computer's art remains to be seen. But this much can surely be said, that their effective use demands a more profound understanding of the mathematics of the problem, and a more detailed acquaintance with the potential sources of error, than is ever required by a computation whose development can be watched, step by step, as it proceeds.

Thus the *analysis* component of 'numerical analysis' is essential. We rely on tools of classical real analysis, such as the notions of continuity, differentiability, Taylor expansion, and convergence of sequences and series. Should you need to improve your analysis background, we recommend

- Walter Rudin, *Principles of Mathematical Analysis*, 3rd ed., McGraw–Hill, New York, 1976.

The methods we study typically require continuous variables to be approximated at finitely many points, that is, *discretized*. Nonlinearities are often finessed by *linearization*. These two compromises reduce a wide range of equations to familiar finite-dimensional, linear algebra problems, and thus we organize our study around a set of fundamental matrix algorithms that we revisit and refine as the semester progresses.

Use the following wonderful books to hone your matrix analysis skills:

- Peter Lax, *Linear Algebra*, Wiley, New York, 1997;
- Carl Meyer, *Applied Matrix Analysis and Linear Algebra*, SIAM, Philadelphia, 2000;
- Gilbert Strang, *Linear Algebra and Its Applications*, 3rd Ed., Harcourt, 1988.

These lecture notes were developed for a course that was supplemented by two texts: *Numerical Linear Algebra* by Trefethen and Bau, and either *Numerical Analysis* by Kincaid and Cheney, or *An Introduction to Numerical Analysis* by Süli and Mayers. These notes have benefited from this pedigree, and thus reflect certain hallmarks of these books. We have also been significantly influenced by G. W. Stewart's inspiring volumes, *Afternotes on Numerical Analysis* and *Afternotes Goes to Graduate School*. I am grateful for comments and corrections from past students, and welcome suggestions for further repair and amendment.