Lecture 14: Matrix Formulation of Spline Interpolants

The last lecture described the construction of spline interpolants as the linear combination of basis functions called B-splines. We spent considerable time studying the construction of these B-splines. Now we will turn our attention to the ultimate goal: construction of the spline interpolants themselves.

2.8.2. Matrix formulation of spline interpolation.

Given a system of knots $\{x_j\}$, we seek the spline S_k that is a degree-k polynomial on each subinterval (x_j, x_{j+1}) with $S_k \in C^{k-1}(\mathbb{R})$ for $k \geq 1$. In terms of the degree-k B-spline basis $\{B_{j,k}\}$, we have

$$S_k(x) = \sum_{j=-\infty}^{\infty} c_{j,k} B_{j,k}(x),$$

with the convention of setting $c_{j,k} = 0$ if $B_{j,k}$ is zero throughout the interval $[x_0, x_n]$. Since $B_{j,k}$ is supported (i.e., nonzero) only on the interval (x_j, x_{j+k+1}), we have

$$S_k(x) = \sum_{j=-k}^{n-1} c_{j,k} B_{j,k}(x),$$

except for $k = 0$, where the limits on the sum go from $j = 0$ to $j = n$.

The case of $k = 0$ (piecewise constant splines) is trivial, as $c_{j,0} = f_j$ for $0 \leq j \leq n$. Hence, from now on assume $k \geq 1$, in which case we have $n + k$ unknowns $\{c_{j,k}\}$ in the above linear combination for S_k. However, the interpolation requirement $S_k(x_j) = f_j$ provides only $n + 1$ constraints.† Overall, $(n + k) - (n + 1) = k - 1$ additional constraints are needed to uniquely specify the spline S_k.

The freedom we have just described must manifest itself in the associated linear algebra. For $\ell = 0, \ldots, n$, the interpolation condition $S_k(x_\ell) = f_\ell$, i.e.,

$$\sum_{j=-k}^{n-1} c_{j,k} B_{j,k}(x_\ell) = f_\ell$$

can be written as a row of the linear system

$$\begin{bmatrix} B_{-k,k}(x_0) & B_{-k+1,k}(x_0) & \cdots & B_{n-1,k}(x_0) \\ B_{-k,k}(x_1) & B_{-k+1,k}(x_1) & \cdots & B_{n-1,k}(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ B_{-k,k}(x_n) & B_{-k+1,k}(x_n) & \cdots & B_{n-1,k}(x_n) \end{bmatrix} \begin{bmatrix} c_{-k,k} \\ c_{-k+1,k} \\ \vdots \\ c_{n-1,k} \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}. \tag{14.1}$$

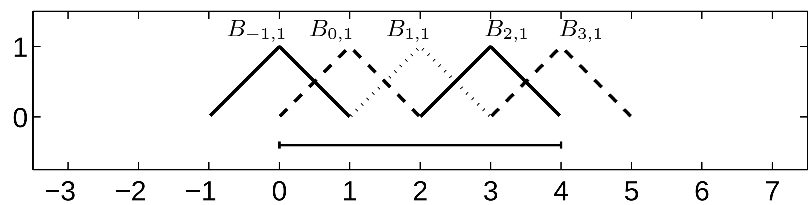
The matrix on the left has $n + 1$ rows and $n + k$ columns.

†What about the requirement $S_k \in C^{k-1}(\mathbb{R})$ that ensures continuity and smoothness? Does it contribute any additional equations that can be used to uniquely determine the $\{c_{j,k}\}$, as was the case with Hermite interpolation? No: By construction, each B-spline satisfies this requirement already, so there are no extra continuity constraints lurking around for us to impose.

Linear splines. To construct the matrix in the linear system (14.1), we must know the value of B-splines at the various knots. When $k = 1$, one can easily see that

$$B_{j,1}(x_\ell) = \begin{cases} 1, & \ell = j+1; \\ 0, & \ell \neq j+1, \end{cases}$$

as confirmed in the following plot of several $B_{j,1}$ for knots $x_j = j$.



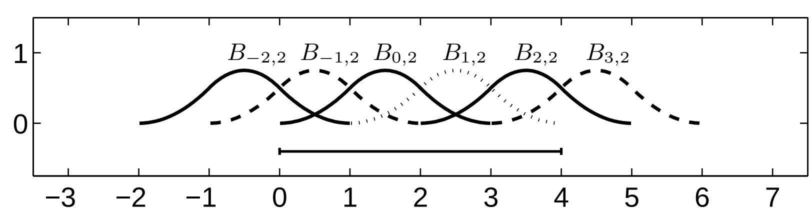
The linear system (14.1) will involve a $(n+1) \times (n+1)$ square matrix; in fact, it takes the trivial form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} c_{-1,1} \\ c_{0,1} \\ \vdots \\ c_{n-1,1} \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix},$$

so the coefficients are simply

$$c_{j-1,1} = f_j, \qquad j = 0, \ldots, n.$$

Quadratic splines. When $k = 2$, the situation becomes more interesting, for now the matrix in equation (14.1) has dimension $(n+1) \times (n+2)$: there are more variables than constraints, reflecting the fact that there are infinitely many quadratic splines that interpolate the data. To determine the matrix entries, first consider the following plot of several quadratic B-splines based again on the knots $x_j = j$.



For simplicity, we shall work out explicit entries in the case that the knots are uniformly spaced ($x_j = x_0 + jh$ for fixed $h > 0$). The recursion that defines quadratic B-splines then takes the form

$$\begin{aligned} B_{j,2}(x) &= \Big(\frac{x - x_j}{x_{j+2} - x_j}\Big) B_{j,1}(x) + \Big(\frac{x_{j+3} - x}{x_{j+3} - x_{j+1}}\Big) B_{j+1,1}(x) \\ &= \frac{1}{2h}(x - x_j) B_{j,1}(x) + \frac{1}{2h}(x_{j+3} - x) B_{j+1,1}(x). \end{aligned}$$

We know by construction that $B_{j,2}(x_\ell) = 0$ unless $\ell = j+1$ or $\ell = j+2$, and in those latter cases

$$B_{j,2}(x_{j+1}) = \frac{1}{2h}(x_{j+1} - x_j)B_{j,1}(x_{j+1}) + \frac{1}{2h}(x_{j+3} - x_{j+1})B_{j+1,1}(x_{j+1}) = \frac{1}{2h}h \cdot 1 + \frac{1}{2h}2h \cdot 0 = 1/2;$$

$$B_{j,2}(x_{j+2}) = \frac{1}{2h}(x_{j+2} - x_j)B_{j,1}(x_{j+2}) + \frac{1}{2h}(x_{j+3} - x_{j+2})B_{j+1,1}(x_{j+2}) = \frac{1}{2h}2h \cdot 0 + \frac{1}{2h}h \cdot 1 = 1/2.$$

In summary,

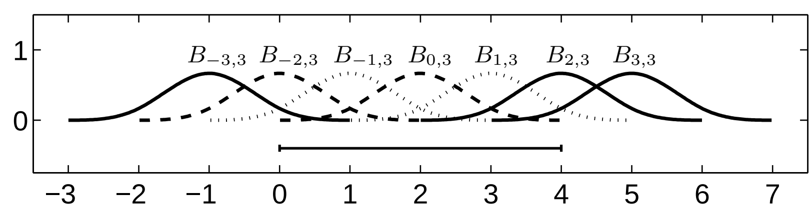
$$B_{j,2}(x_\ell) = \begin{cases} 1/2, & \ell = j+1 \\ 1/2, & \ell = j+2 \\ 0, & \ell \notin \{j+1, j+2\}. \end{cases}$$

Thus, the linear system (14.1) takes the form

$$\begin{bmatrix} 1/2 & 1/2 & & & \\ & 1/2 & 1/2 & & \\ & & \ddots & \ddots & \\ & & & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} c_{-2,2} \\ c_{-1,2} \\ c_{0,2} \\ \vdots \\ c_{n-1,2} \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix},$$

where the blank entries are zero. To make this a well-determined system, one needs to add one more constraint. Effectively, this appends a new row to the matrix and a new entry to the right hand side vector. The operation is very similar to the more important case of cubic splines, which we will now work through in detail.

Cubic splines. This famous case with $k = 3$ is a little more intricate. In this case, the matrix in (14.1) has $n+1$ rows but $n+3$ columns, so we need to impose two additional constraints. Recall that the cubic B-splines take the following form, where once again $x_j = j$.



Assuming again that the knots are uniformly spaced, $x_j = x_0 + jh$, then with a little calculation one can confirm that

$$B_{j,3}(x_\ell) = \begin{cases} \alpha, & \ell = j+1 \\ \beta, & \ell = j+2 \\ \alpha, & \ell = j+3 \\ 0, & \ell \notin \{j+1, j+2, j+3\}. \end{cases}$$

where α and β are simple constants whose computation is left as an exercise for the reader.‡

‡Use the recurrence that defines the B-splines, together with the known values of $B_{j,2}(x_\ell)$ given above. More labor is required to compute the value of $B_{j,3}(x)$ when x is not a knot, but remember that you do not need to know these intermediate values to set up the linear system.

The linear system (14.1) now takes the form

$$\begin{bmatrix} \alpha & \beta & \alpha & & & \\ & \alpha & \beta & \alpha & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha & \beta & \alpha \end{bmatrix} \begin{bmatrix} c_{-3,3} \\ c_{-2,3} \\ c_{-1,3} \\ \vdots \\ c_{n-1,3} \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}, \qquad (14.2)$$

where the blank entries are zero.

To uniquely determine the spline coefficients, we impose the *natural spline* conditions, $S_3''(x_0) = S_3''(x_n) = 0$. With considerable tedious labor, one can verify that these conditions reduce to

$$\begin{aligned} (x_2 - x_{-1})c_{-3,3} - (x_2 + x_1 - x_{-1} - x_{-2})c_{-2,3} + (x_1 - x_{-2})c_{-1,3} &= 0 \\ (x_{n+2} - x_{n-1})c_{n-3,3} - (x_{n+2} + x_{n+1} - x_{n-1} - x_{n-2})c_{n-2,3} + (x_{n+1} - x_{n-2})c_{n-1,3} &= 0. \end{aligned}$$

If the knots are equally spaced ($x_j = x_0 + jh$) these conditions simplify to

$$\begin{aligned} 3hc_{-3,3} - 6hc_{-2,3} + 3hc_{-1,3} &= 0 \\ 3hc_{n-3,3} - 6hc_{n-2,3} + 3hc_{n-1,3} &= 0. \end{aligned}$$

Dividing these equations by $h > 0$,

$$\begin{aligned} 3c_{-3,3} - 6c_{-2,3} + 3c_{-1,3} &= 0 \\ 3c_{n-3,3} - 6c_{n-2,3} + 3c_{n-1,3} &= 0. \end{aligned}$$

We wish to augment the linear system (14.2) with these two equations. It is conventional to insert the natural spline conditions at x_0 and x_n in the first and last rows of the new system, respectively. This gives

$$\begin{bmatrix} 3 & -6 & 3 & & & \\ \alpha & \beta & \alpha & & & \\ & \alpha & \beta & \alpha & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha & \beta & \alpha \\ & & & 3 & -6 & 3 \end{bmatrix} \begin{bmatrix} c_{-3,3} \\ c_{-2,3} \\ c_{-1,3} \\ \vdots \\ c_{n-2,3} \\ c_{n-1,3} \end{bmatrix} = \begin{bmatrix} 0 \\ f_0 \\ f_1 \\ \vdots \\ f_n \\ 0 \end{bmatrix},$$

which is now a square $(n+3) \times (n+3)$ linear system that one can solve to obtain the unique coefficients $\{c_{j,3}\}$.