Lecture 15: Trigonometric Interpolation

2.9 Trigonometric Interpolation.

Thus far all our interpolation schemes have been based on polynomials. However, if the function f is *periodic*, one might naturally prefer to interpolate f with some set of periodic functions.

To be concrete, suppose we have a continuous 2π-periodic[†] function f that we wish to interpolate at the uniformly spaced points $x_k = 2\pi k/n$ for $k = 0, \ldots, n$ with $n = 5$. The interpolant will be built as a linear combination of the 2π-periodic functions

$$b_0(x) = 1, \quad b_1(x) = \sin(x), \quad b_2(x) = \cos(x), \quad b_3(x) = \sin(2x), \quad b_4(x) = \cos(2x).$$

Note that we have *six* interpolation conditions at x_k for $k = 0, \ldots, 5$, but only *five* basis functions. This is not a problem: since f is periodic, $f(x_0) = f(x_n)$, and the same will be true of our 2π-periodic interpolant: the last interpolation condition is automatically satisfied.

We shall construct an interpolant of the form

$$t_n(x) = \sum_{k=0}^{n-1} c_k b_k(x)$$

such that

$$t_n(x_j) = f(x_j), \quad j = 0, \ldots, n-1.$$

To compute the unknown coefficients $c_0, \ldots, c_5$, we set up a linear system as usual,

$$\begin{bmatrix} b_0(x_0) & b_1(x_0) & b_2(x_0) & b_3(x_0) & b_4(x_0) \\ b_0(x_1) & b_1(x_1) & b_2(x_1) & b_3(x_1) & b_4(x_1) \\ b_0(x_2) & b_1(x_2) & b_2(x_2) & b_3(x_2) & b_4(x_2) \\ b_0(x_3) & b_1(x_3) & b_2(x_3) & b_3(x_3) & b_4(x_3) \\ b_0(x_4) & b_1(x_4) & b_2(x_4) & b_3(x_4) & b_4(x_4) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \end{bmatrix},$$

which can be readily generalized to accommodate more interpolation points. As a practical matter, one might wonder how accurately this system can be solved. What can be said of the *conditioning* of the matrix?

Rather than investigating this question directly, we shall first transform to a slightly more convenient basis. Recall Euler's formula,

$$\mathrm{e}^{\mathrm{i}\theta x} = \cos(\theta x) + \mathrm{i}\sin(\theta x),$$

which also implies that

$$\mathrm{e}^{-\mathrm{i}\theta x} = \cos(\theta x) - \mathrm{i}\sin(\theta x).$$

From this it follows that

$$\mathrm{span}\{\mathrm{e}^{\mathrm{i}\theta x}, \mathrm{e}^{-\mathrm{i}\theta x}\} = \{\cos(\theta x), \sin(\theta x)\}.$$

Note that we can also write $b_0(x) \equiv 1 = \mathrm{e}^{\mathrm{i}0x}$. Putting these pieces together, we arrive at an alternative basis:

$$\mathrm{span}\{1, \sin(x), \cos(x), \sin(2x), \cos(2x)\} = \mathrm{span}\{\mathrm{e}^{-2\mathrm{i}x}, \mathrm{e}^{-\mathrm{i}x}, \mathrm{e}^{0\mathrm{i}x}, \mathrm{e}^{\mathrm{i}x}, \mathrm{e}^{2\mathrm{i}x}\}.$$

[†] This means that f is continuous throughout $\mathbb{R}$ and $f(x) = f(x + 2\pi)$ for all $x \in \mathbb{R}$.

We shall thus write the interpolant t_n in the form

$$t_n(x) = \sum_{k=-2}^{2} \gamma_k \mathrm{e}^{\mathrm{i}kx} = \sum_{k=-2}^{2} \gamma_k (\mathrm{e}^{\mathrm{i}x})^k.$$

This last sum is written in a manner that emphasizes that t_n is *a polynomial in the variable* $\mathrm{e}^{\mathrm{i}x}$, and hence we call t_n a *trigonometric polynomial.* In this basis, the interpolation conditions give the linear system

$$\begin{bmatrix} \mathrm{e}^{-2\mathrm{i}x_0} & \mathrm{e}^{-\mathrm{i}x_0} & \mathrm{e}^{0\mathrm{i}x_0} & \mathrm{e}^{\mathrm{i}x_0} & \mathrm{e}^{\mathrm{i}2x_0} \\ \mathrm{e}^{-2\mathrm{i}x_1} & \mathrm{e}^{-\mathrm{i}x_1} & \mathrm{e}^{0\mathrm{i}x_1} & \mathrm{e}^{\mathrm{i}x_1} & \mathrm{e}^{\mathrm{i}2x_1} \\ \mathrm{e}^{-2\mathrm{i}x_2} & \mathrm{e}^{-\mathrm{i}x_2} & \mathrm{e}^{0\mathrm{i}x_2} & \mathrm{e}^{\mathrm{i}x_2} & \mathrm{e}^{\mathrm{i}2x_2} \\ \mathrm{e}^{-2\mathrm{i}x_3} & \mathrm{e}^{-\mathrm{i}x_3} & \mathrm{e}^{0\mathrm{i}x_3} & \mathrm{e}^{\mathrm{i}x_3} & \mathrm{e}^{\mathrm{i}2x_3} \\ \mathrm{e}^{-2\mathrm{i}x_4} & \mathrm{e}^{-\mathrm{i}x_4} & \mathrm{e}^{0\mathrm{i}x_4} & \mathrm{e}^{\mathrm{i}x_4} & \mathrm{e}^{\mathrm{i}2x_4} \end{bmatrix} \begin{bmatrix} \gamma_{-2} \\ \gamma_{-1} \\ \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \end{bmatrix},$$

again with the natural generalization to larger odd integers n. At first blush this matrix looks no simpler than the one we first encountered, but there is fascinating structure here. Notice that a generic entry of this matrix has the form $\mathrm{e}^{\ell \mathrm{i}x_k}$ for $\ell = -(n-1)/2, \ldots, (n-1)/2$ and $k = 0, \ldots, n-1$. Since $x_k = 2\pi k/n$, we can rewrite this entry as

$$\mathrm{e}^{\ell \mathrm{i}x_k} = (\mathrm{e}^{\mathrm{i}x_k})^\ell = (\mathrm{e}^{2\pi \mathrm{i}k/n})^\ell = (\mathrm{e}^{2\pi\mathrm{i}/n})^{k\ell} = \omega^{k\ell},$$

where $\omega = \mathrm{e}^{2\pi\mathrm{i}/n}$ is an *nth root of unity* (since $\omega^n = 1$). In the $n = 5$ case, the linear system can thus be written as

$$\begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^{-2} & \omega^{-1} & \omega^0 & \omega^1 & \omega^2 \\ \omega^{-4} & \omega^{-2} & \omega^0 & \omega^2 & \omega^4 \\ \omega^{-6} & \omega^{-3} & \omega^0 & \omega^3 & \omega^6 \\ \omega^{-8} & \omega^{-4} & \omega^0 & \omega^4 & \omega^8 \end{bmatrix} \begin{bmatrix} \gamma_{-2} \\ \gamma_{-1} \\ \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \end{bmatrix}.$$

Denote this system by $\mathbf{Fg} = \mathbf{f}$. Notice that each column of $\mathbf{F}$ equals some (entrywise) power of the vector

$$\begin{bmatrix} \omega^0 \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{bmatrix}.$$

In other words, *the matrix has Vandermonde structure*! From our past experience, we might well expect such a matrix to be highly ill-conditioned. Before jumping to this conclusion, we shall examine $\mathbf{F}^*\mathbf{F}$. To form $\mathbf{F}^*$ (the conjugate-transpose of $\mathbf{F}$), we note that $\overline{\omega^{-\ell}} = \omega^\ell$, so

$$\mathbf{F}^*\mathbf{F} = \begin{bmatrix} \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^8 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 \\ \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} \\ \omega^0 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \omega^{-8} \end{bmatrix} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^{-2} & \omega^{-1} & \omega^0 & \omega^1 & \omega^2 \\ \omega^{-4} & \omega^{-2} & \omega^0 & \omega^2 & \omega^4 \\ \omega^{-6} & \omega^{-3} & \omega^0 & \omega^3 & \omega^6 \\ \omega^{-8} & \omega^{-4} & \omega^0 & \omega^4 & \omega^8 \end{bmatrix}.$$

The (ℓ, k) entry for $\mathbf{F}^*\mathbf{F}$ thus takes the form

$$(\mathbf{F}^*\mathbf{F})_{\ell,k} = \omega^0 + \omega^{(k-\ell)} + \omega^{2(k-\ell)} + \omega^{3(k-\ell)} + \omega^{4(k-\ell)}.$$

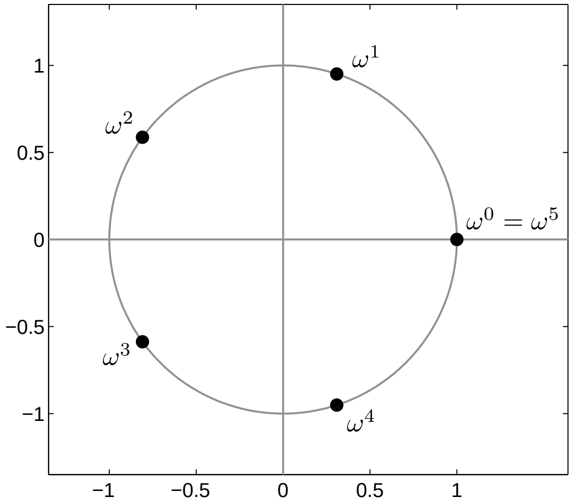
On the diagonal, when $\ell = k$, we simply have

$$(\mathbf{F}^*\mathbf{F})_{k,k} = \omega^0 + \omega^0 + \omega^0 + \omega^0 + \omega^0 = n.$$

On the off-diagonal, use $\omega^n = 1$ to see that all the off diagonal entries simplify to

$$(\mathbf{F}^*\mathbf{F})_{\ell,k} = \omega^0 + \omega^1 + \omega^2 + \omega^3 + \omega^4, \qquad \ell \neq k.$$

You can think of this last entry as n times the average of ω^0, ω^1, ω^2, ω^3, and ω^4, which are uniformly spaced points on the unit circle, shown in the plot below.



As these points are uniformly positioned about the unit circle, their mean must be zero, and hence

$$(\mathbf{F}^*\mathbf{F})_{\ell,k} = 0, \qquad \ell \neq k.$$

We have arrived at the conclusion that

$$\mathbf{F}^*\mathbf{F} = n\mathbf{I},$$

thus giving a formula for the inverse:

$$\mathbf{F}^{-1} = \frac{1}{n}\mathbf{F}^*.$$

The system $\mathbf{F}\mathbf{g} = \mathbf{f}$ can be immediately solved without the need for any factorization of $\mathbf{F}$:

$$\mathbf{g} = \frac{1}{n}\mathbf{F}^*\mathbf{f}.$$

The ready formula for $\mathbf{F}^{-1}$ is reminiscent of a unitary matrix. (Recall that $\mathbf{Q} \in \mathbb{C}^{n\times n}$ is unitary if and only if $\mathbf{Q}^{-1} = \mathbf{Q}^*$.) In fact, we see that the matrices

$$\frac{1}{\sqrt{n}}\mathbf{F} \quad \text{and} \quad \frac{1}{\sqrt{n}}\mathbf{F}^*$$

are indeed unitary, and hence $\|n^{-1/2}\mathbf{F}\|_2 = \|n^{-1/2}\mathbf{F}^*\|_2 = 1$. From this we can compute the condition number of $\mathbf{F}$:

$$\|\mathbf{F}\|_2\|\mathbf{F}^{-1}\|_2 = \frac{1}{n}\|\mathbf{F}\|_2\|\mathbf{F}^*\|_2 = \|n^{-1/2}\mathbf{F}\|_2\|n^{-1/2}\mathbf{F}^*\|_2 = 1.$$

This special Vandermonde matrix is perfectly conditioned! The key distinction between this case and standard polynomial interpolation is that now we have a Vandermonde matrix based on *points* e^{ix_k} *that are equally spaced about the unit circle in the complex plane*, whereas before our points were distributed over an interval of the real line. This distinction makes all the difference between an unstable system and one that is not only perfectly stable, but also forms the cornerstone of modern signal processing.

In fact, we have just computed the 'Discrete Fourier Transform' (DFT) of the data vector

$$\begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{n-1}) \end{bmatrix}.$$

The coefficients $\gamma_{-(n-1)/2}, \ldots, \gamma_{(n-1)/2}$ that make up the vector $\mathbf{g} = n^{-1}\mathbf{F}^*\mathbf{f}$ are the *discrete Fourier coefficients* of the data in $\mathbf{f}$. Normally we would require $O(n^2)$ operations to compute these coefficients using matrix-vector multiplication with $\mathbf{F}^*$, but Cooley and Tukey discovered in 1965 that given the amazing structure in $\mathbf{F}^*$, one can arrange operations so as to compute $\mathbf{g} = n^{-1}\mathbf{F}^*\mathbf{f}$ in only $O(n \log n)$ operations – a procedure (apparently known earlier to Gauss) that we now famously call the *Fast Fourier Transform* (FFT).

We can thus summarize this lecture as follows: the FFT of a vector of uniform samples of a 2π-periodic function f is simply the set of coefficients for the trigonometric polynomial interpolant to f at those sample points.