Lecture 22: Minimax Approximation, Optimal Interpolation, Chebyshev Polynomials

3.4.2. Optimal interpolation points.

As an application of the minimax approximation procedure, we consider how best to choose interpolation points $\{x_j\}_{j=0}^n$ to minimize

$$\|f - p_n\|_{L^\infty},$$

where $p_n \in \mathcal{P}_n$ is the interpolant to f at the specified points.

Recall the interpolation error bound developed in Section 2.4: If $f \in C^{n+1}[a,b]$, then for any $x \in [a,b]$ there exists some $\xi \in [a,b]$ such that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{j=0}^n (x - x_j).$$

Taking absolute values and maximizing over $[a,b]$ yields the bound

$$\|f - p_n\|_{L^\infty} = \max_{\xi\in[a,b]} \frac{|f^{(n+1)}(\xi)|}{(n+1)!} \max_{x\in[a,b]} \Big|\prod_{j=0}^n (x - x_j)\Big|.$$

For Runge's example, $f(x) = 1/(1+x^2)$ for $x \in [-5,5]$, we observed that $\|f - p_n\|_{L^\infty} \to \infty$ as $n \to \infty$ if the interpolation points $\{x_j\}$ are uniformly spaced over $[-5,5]$. However, Marcinkiewicz's theorem (Section 2.4) guarantees there is always some scheme for assigning the interpolation points such that $\|f - p_n\|_{L^\infty} \to 0$ as $n \to \infty$. In the case of Runge's function, we observed that the choice

$$x_j = 5\cos(j\pi/n), \qquad j = 0, \ldots, n$$

is one such scheme. While there is no fail-safe *a priori* system for picking interpolations points that will yield uniform convergence for *all* $f \in C[a,b]$, there is a distinguished choice that works exceptionally well for just about every function you will encounter in practice. We determine this set of interpolation points by choosing those $\{x_j\}_{j=0}^n$ that *minimize the error bound* (which is distinct from – but hopefully akin to – minimizing the error itself, $\|f - p_n\|_{L^\infty}$). That is, we want to solve

$$\min_{x_0,\ldots,x_n} \max_{x\in[a,b]} \Big|\prod_{j=0}^n (x - x_j)\Big|. \tag{22.1}$$

Notice that

$$\begin{aligned}\prod_{j=0}^n (x - x_j) &= x^{n+1} - x^n \sum_{j=0}^n x_j + x^{n-1}\sum_{j=0}^n\sum_{k=0}^n x_j x_k - \cdots + (-1)^{n+1}\prod_{j=0}^n x_j\\ &= x^{n+1} - r(x),\end{aligned}$$

where $r \in \mathcal{P}_n$ is a degree-n polynomial depending on the interpolation nodes $\{x_j\}_{j=0}^n$.

To find the optimal interpolation points according to (22.1), we should solve

$$\min_{r\in\mathcal{P}_n} \max_{x\in[a,b]} |x^{n+1} - r(x)| = \min_{r\in\mathcal{P}_n} \|x^{n+1} - r(x)\|_{L^\infty}.$$

Here the goal is to approximate an $(n+1)$-degree polynomial, x^{n+1}, with an n-degree polynomial. The method of solution is somewhat indirect: we will produce a class of polynomials of the form $x^{n+1} - r(x)$ that satisfy the requirements of the Oscillation Theorem, and thus $r(x)$ must be the minimax polynomial approximation to x^{n+1}. As we shall see, the roots of the resulting polynomial $x^{n+1} - r(x)$ will fall in the interval $[a, b]$, and can thus be regarded as 'optimal' interpolation points. For simplicity, we shall focus on the interval $[a, b] = [-1, 1]$.

Definition. The degree-n *Chebyshev polynomial* is defined for $x \in [-1, 1]$ by the formula

$$T_n(x) = \cos(n \cos^{-1} x).$$

At first glance, this formula may not appear to define a polynomial at all, since it involves trigonometric functions.[†] But computing the first few examples, we find

$$\begin{aligned} n = 0: &\quad T_0(x) = \cos(0 \cos^{-1} x) = \cos(0) = 1 \\ n = 1: &\quad T_1(x) = \cos(\cos^{-1} x) = x \\ n = 2: &\quad T_2(x) = \cos(2 \cos^{-1} x) = 2\cos^2(\cos^{-1} x) - 1 = 2x^2 - 1. \end{aligned}$$

For $n = 2$, we employed the identity $\cos 2\theta = 2\cos^2\theta - 1$, substituting $\theta = \cos^{-1} x$. More generally, we have the identity

$$\cos(n+1)\theta = 2\cos\theta\cos n\theta - \cos(n-1)\theta.$$

This formula implies, for $n \geq 2$,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

a formula related to the three term recurrence used to construct orthogonal polynomials. (In fact, Chebyshev polynomials are orthogonal polynomials on $[-1, 1]$ with respect to the inner product $\langle f, g\rangle = \int_a^b f(x)g(x)(1-x^2)^{-1/2}$, a fact we will use when studying Gaussian quadrature in a few lectures.)

Chebyshev polynomials exhibit a wealth of interesting properties, of which we mention just three.

Proposition. Let T_n be the degree-n Chebyshev polynomial, $T_n(x) = \cos(n \cos^{-1} x)$ for $x \in [-1, 1]$.

- $|T_n(x)| \leq 1$ for $x \in [-1, 1]$.
- The roots of T_n are the n points $\xi_j = \cos\frac{(2j-1)\pi}{2n}$, $j = 1, \ldots, n$.
- For $n \geq 1$, $|T_n(x)|$ is maximized on $[-1, 1]$ at the $n+1$ points $\eta_j = \cos(j\pi/n)$, $j = 0, \ldots, n$:

$$T_n(\eta_j) = (-1)^j.$$

Proof. These results follow from direct calculations. For $x \in [-1, 1]$, $T_n(x) = \cos(n\cos^{-1}(x))$ cannot exceed one in magnitude because cosine cannot exceed one in magnitude. To verify the formula for the roots, compute

$$T_n(\xi_j) = \cos\left(n\cos^{-1}\cos\left(\frac{(2j-1)\pi}{2n}\right)\right) = \cos\left(\frac{(2j-1)\pi}{2}\right) = 0,$$

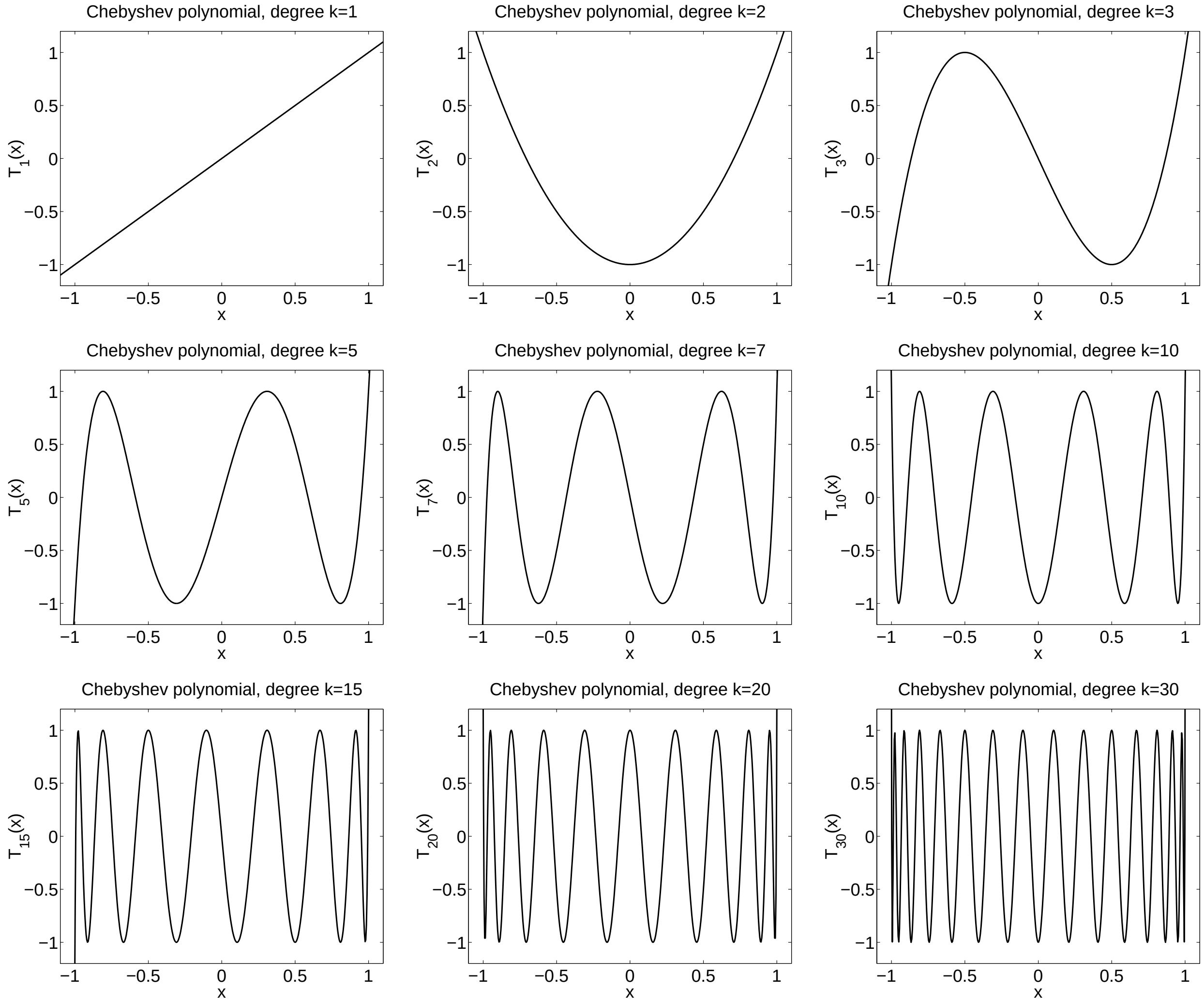
[†] Furthermore, it doesn't apply if $|x| > 1$. For such x one can define the Chebyshev polynomials using hyperbolic trigonometric functions, $T_n(x) = \cosh(n\cosh^{-1} x)$. Indeed, using hyperbolic trigonometric identities, one can show that this expression generates for $x \notin [-1, 1]$ the same polynomials we get for $x \in [-1, 1]$ from the standard trigonometric identities.

since cosine is zero at half-integer multiples of π. Similarly,

$$T_n(\eta_j) = \cos\left(n\cos^{-1}\cos\left(\frac{j\pi}{n}\right)\right) = \cos(j\pi) = (-1)^j.$$

Since $T_n(\eta_j)$ is a nonzero degree-n polynomial, it cannot attain more than $n+1$ extrema on $[-1,1]$, including the endpoint: we have thus characterized all the maxima of $|T_n|$ on $[-1,1]$. $\blacksquare$.

The figures below show Chebyshev polynomials T_n for nine different values of n.



Chebyshev polynomial, degree k=1

Chebyshev polynomial, degree k=2

Chebyshev polynomial, degree k=3

Chebyshev polynomial, degree k=5

Chebyshev polynomial, degree k=7

Chebyshev polynomial, degree k=10

Chebyshev polynomial, degree k=15

Chebyshev polynomial, degree k=20

Chebyshev polynomial, degree k=30

The punchline. Finally, we are ready to solve the key minimax problem that will reveal optimal interpolation points. Looking at the above plots of Chebyshev polynomials, with their striking equioscillation properties, perhaps you have already guessed the solution yourself.

We defined the Chebyshev polynomials so that

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

with $T_0(x) = 1$ and $T_1(x) = x$. Thus T_{n+1} has the leading coefficient 2^n for $n \ge 0$. Define

$$\widehat{T}_{n+1} = 2^{-n}T_{n+1}$$

for $n \geq 0$, with $\widehat{T}_0(x) = 1$. These *normalized* Chebyshev polynomials are *monic*, i.e., the leading term in $\widehat{T}_{n+1}(x)$ is x^{n+1}, rather than $2^n x^{n+1}$ as for $T_{n+1}(x)$. Thus, we can write

$$\widehat{T}_{n+1}(x) = x^{n+1} - r_n(x)$$

for some polynomial $r_n(x) = x^{n+1} - \widehat{T}_{n+1}(x) \in \mathcal{P}_n$. We do not especially care about the particular coefficients of this r_n; our quarry will be the *roots* of $\widehat{T}_{n+1}$, the optimal interpolation points.

For $n \geq 0$, the polynomials $\widehat{T}_{n+1}(x)$ oscillate between $\pm 2^{-n}$ for $x \in [-1, 1]$, with the maximal values attained at

$$\eta_j = \cos\left(\frac{j\pi}{n+1}\right)$$

for $j = 0, \ldots, n+1$. In particular,

$$\widehat{T}_{n+1}(\eta_j) = (\eta_j)^{n+1} - r_n(\eta_j) = (-1)^j 2^{-n}.$$

Thus, we have found a polynomial $r_n \in \mathcal{P}_n$, together with $n+2$ distinct points, $\eta_j \in [-1, 1]$ where the maximum error

$$\max_{x \in [-1,1]} |x^{n+1} - r_n(x)| = 2^{-n}$$

is attained with alternating sign. Thus, by the oscillation theorem, we have found the minimax approximation to x^{n+1}.

Theorem (Optimal approximation of x^{n+1}). The optimal approximation to x^{n+1} from $\mathcal{P}_n$ on the interval $x \in [-1, 1]$ is given by

$$r_n(x) = x^{n+1} - \widehat{T}_{k+1}(x) = x^{n+1} - 2^{-n} T_{k+1}(x) \in \mathcal{P}_n.$$

Thus, the optimal interpolation points are those $n+1$ roots of $x^{n+1} - r_n(x)$, that is, the roots of the degree-$(n+1)$ Chebyshev polynomial:

$$\xi_j = \cos\left(\frac{(2j+1)\pi}{2n+2}\right), \quad j = 0, \ldots, n.$$

For generic intervals $[a, b]$, a change of variable demonstrates that the same points, appropriately shifted and scaled, will be optimal.

Similar properties hold if interpolation is performed at the $n+1$ points

$$\eta_j = \cos\left(\frac{j\pi}{n}\right), \quad j = 0, \ldots, n,$$

which are also called Chebyshev points and are perhaps more popular due to their slightly simpler formula. (We used these points to successfully interpolate Runge's function, scaled to the interval $[-5, 5]$.) While these points differ from the roots of the Chebyshev polynomial, they *have the same distribution* as $n \to \infty$. That is the key.