

### Lecture 23b: Peano Kernel Analysis

In the last lecture we proved an error bound for the trapezoid rule using the mean value theorem for integrals, and we stated a result for Simpson's rule without proof. In this lecture, we present a general convergence theory that is applicable to a wide range of quadrature rules.

**4.2. Error analysis for quadrature rules.** Consider a general quadrature rule of the form

$$I(f) = \sum_{j=0}^m w_j f(x_j)$$

for distinct nodes  $\{x_j\}_{j=0}^m \subset [a, b]$  and weights  $\{w_j\}_{j=0}^m$ .<sup>†</sup> All Newton–Cotes methods fit this template. For example, the trapezoid rule has  $x_0 = a$ ,  $x_1 = b$ , and  $w_0 = w_1 = (b - a)/2$ .

Define the error function for this quadrature rule as

$$E(f) = \int_a^b f(x) dx - I(f).$$

The integral is linear ( $\int f + g = \int f + \int g$ ), and so we expect the quadrature rule  $I(f)$  to share this property. Indeed it does, and thus the error function  $E(f)$  is also linear:

$$\begin{aligned} E(\alpha f + g) &= \int_a^b \alpha f(x) + g(x) dx - \sum_{j=0}^m w_j (\alpha f(x_j) + g(x_j)) \\ &= \alpha \left( \int_a^b f(x) dx - \sum_{j=0}^m w_j f(x_j) \right) + \left( \int_a^b g(x) dx - \sum_{j=0}^m w_j g(x_j) \right) \\ &= \alpha E(f) + E(g) \end{aligned}$$

for arbitrary scalar  $\alpha$  and functions  $f$  and  $g$ .

Our goal is to describe  $E(f)$  using basic properties of the function  $f$  and the particular quadrature rule  $I(f)$ . We shall develop a general theory, then apply it to specific quadrature rules.

The analysis to follow makes a key assumption: We must know *a priori* that the quadrature rule  $I$  *exactly integrates all polynomials of degree  $n$  or less*:

$$\text{If } p \in \mathcal{P}_n, \text{ then } I(p) = \int_a^b p(x) dx.$$

Given a rule, one can find the largest such value of  $n$  by applying the rule to monomials of increasing degree, and comparing to the true integral. For the trapezoid rule, we have  $n = 1$ ; for Simpson's rule,  $n = 3$ .

Suppose further that  $f \in C^{n+1}[a, b]$ , so that we can expand  $f$  in a Taylor series,

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)(x-a)^k}{k!} + \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x-t)^n dt.$$

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<sup>†</sup>Some quadrature rules require evaluations of derivatives of  $f$ , such as  $f'(x_j)$  or  $f''(x_j)$ , etc., as would occur if you integrate the Hermite polynomial interpolant. The theory we discuss here also works in that case, though for simplicity we do not discuss those details.

Here we have used the *integral remainder term* for the Taylor series.<sup>‡</sup>

The first  $n$  terms of the Taylor series form a polynomial of degree  $n$  (or less) in  $x$ , which we shall call  $p_n$ ; denote the remainder term by  $r_n$ :

$$p_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)(x-a)^k}{k!} \in \mathcal{P}_n, \quad r_n(x) = \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x-t)^n dt,$$

and so  $f(x) = p_n(x) + r_n(x)$ . Since  $p_n$  is a polynomial of degree  $n$  or less, it is exactly integrated by the quadrature rule, so  $E(p_n) = 0$ . Using the linearity of  $E$ , we obtain

$$E(f) = E(p_n + r_n) = E(p_n) + E(r_n) = E(r_n).$$

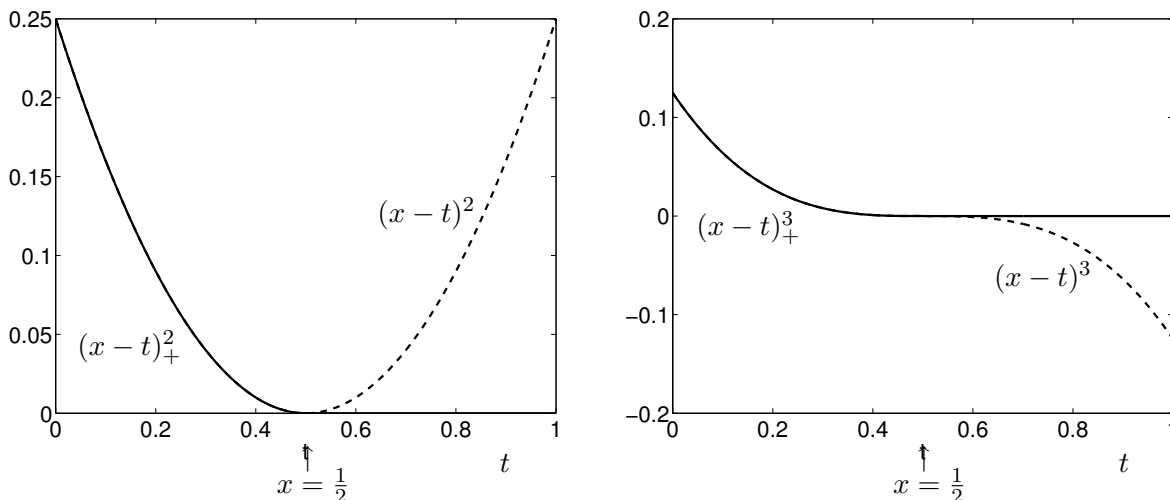
Thus, to describe  $E(f)$  we need only derive an expression for

$$E(r_n) = \int_a^b r_n(x) dx - I(r_n) = \frac{1}{n!} \int_a^b \int_a^x f^{(n+1)}(t)(x-t)^n dt dx - I(r_n).$$

It will be convenient to remove the  $x$  from the upper limit of the interior integral above. Toward this end we introduce the *truncated power function*

$$(x-t)_+^n = \begin{cases} (x-t)^n, & t \leq x; \\ 0, & t > x. \end{cases}$$

Below we compare  $(x-t)_+^2$  (left) and  $(x-t)_+^3$  (right) to standard power functions as a function of  $t$  for fixed  $x = 1/2$ .



Since  $(x-t)_+^n = 0$  for  $t > x$ , we have

$$r_n(x) = \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x-t)^n dt = \frac{1}{n!} \int_a^b f^{(n+1)}(t)(x-t)_+^n dt,$$

and so

$$E(r_n) = E\left(\frac{1}{n!} \int_a^b f^{(n+1)}(t)(x-t)_+^n dt\right) = \frac{1}{n!} E\left(\int_a^b f^{(n+1)}(t)(x-t)_+^n dt\right).$$

<sup>‡</sup>Details about this integral remainder formula can be found in analysis texts, e.g., Strichartz, *The Way of Analysis*, pp. 210-211.

Now we simplify this expression for  $E(r_n)$  even further:

$$\begin{aligned}
 E\left(\int_a^b f^{(n+1)}(t)(x-t)_+^n dt\right) &= \int_a^b \int_a^b f^{(n+1)}(t)(x-t)_+^n dt dx - I\left(\int_a^b f^{(n+1)}(t)(x-t)_+^n dt\right) \\
 &= \int_a^b \int_a^b f^{(n+1)}(t)(x-t)_+^n dt dx - \sum_{j=0}^m w_j \int_a^b f^{(n+1)}(t)(x_j-t)_+^n dt \\
 &= \int_a^b f^{(n+1)}(t) \int_a^b (x-t)_+^n dx dt - \int_a^b f^{(n+1)}(t) \sum_{j=0}^m w_j (x_j-t)_+^n dt \\
 &= \int_a^b f^{(n+1)}(t) \left( \int_a^b (x-t)_+^n dx - \sum_{j=0}^m w_j (x_j-t)_+^n \right) dt \\
 &= \int_a^b f^{(n+1)}(t) E((x-t)_+^n) dt.
 \end{aligned}$$

We have just proved the *Peano kernel theorem*. (In fact, the full theorem is a bit more general than what we proved here, though our development is sufficient for analysis of Newton–Cotes rules.)

**Theorem (Peano kernel theorem).** Suppose  $f \in C^{n+1}[a, b]$  and let  $I(f) = \sum_{j=0}^m w_j f(x_j)$  be a quadrature rule that exactly integrates all polynomials of degree  $n$  or less on  $[a, b]$ . Then

$$E(f) = \frac{1}{n!} \int_a^b f^{(n+1)}(t) K(t) dt,$$

where  $K(t) = E((x-t)_+^n)$ .

This may result may appear somewhat ponderous, but it captures a beautiful idea: the error in the quadrature of  $f$  is related to the integral of a canonical function  $K(t)$  that is independent of  $f$ . In typical circumstances,  $K(t)$  does not change sign on  $[a, b]$ , so the mean value theorem for integrals can be applied to extract  $f$  from inside the integral. This allows  $E(f)$  to be described as a constant (that depends upon  $f$ ) times the integral of the error of the kernel function.

**Corollary.** Provided  $K(t)$  does not change sign on  $[a, b]$ , there exists some  $\xi \in [a, b]$  such that

$$E(f) = \frac{f^{(n+1)}(\xi)}{n!} \int_a^b K(t) dt.$$

This corollary follows from application of the mean value theorem for integrals.

To appreciate how the Peano kernel theorem operates in practice, we shall work several examples.

**Application: trapezoid rule.** Recall that the formula for the trapezoid rule is

$$I(f) = \frac{b-a}{2}(f(a) + f(b)).$$

This quadrature rule exactly integrates linear polynomials, so take  $n = 1$ . In this case,

$$K(t) = E((x-t)_+) = \int_a^b (x-t)_+ dx - I((x-t)_+)$$

$$\begin{aligned}
&= \int_t^b (x-t) dx - I((x-t)_+) \\
&= \left[ \frac{(x-t)^2}{2} \right]_{x=t}^b - \frac{b-a}{2} ((a-t)_+ + (b-t)_+) \\
&= \frac{(b-t)^2}{2} - \frac{(b-a)(b-t)}{2} = \frac{(b-t)(a-t)}{2},
\end{aligned}$$

where we have used the fact that  $(a-t)_+ = 0$  since  $a \leq t$  for all  $t \in [a, b]$ , and  $(b-t)_+ = b-t$  since  $b \geq t$  for all  $t \in [a, b]$ .

Note that for  $t \in [a, b]$ ,  $b-t \geq 0$  and  $a-t \leq 0$ , so  $K(t) \leq 0$  for all  $t \in [a, b]$ . In particular,  $K(t)$  does not change signs on  $[a, b]$ , so we can apply the mean value theorem for integrals, as in the last lecture.

Thus for the trapezoid rule, we have

$$\begin{aligned}
E(f) &= \frac{1}{1!} \int_a^b f''(t) K(t) dt \\
&= f''(\xi) \int_a^b K(t) dt
\end{aligned}$$

for some  $\xi \in [a, b]$ . Furthermore, it is simple to verify that

$$\int_a^b K(t) dt = \int_a^b \frac{(b-t)(a-t)}{2} dt = -\frac{1}{12}(b-a)^3.$$

Thus, we conclude that the error in the trapezoid rule is

$$E(f) = -\frac{1}{12} f''(\xi)(b-a)^3,$$

exactly the same result we derived in the previous lecture via simpler means. The real power of the Peano kernel approach becomes apparent when studying higher-order quadrature formulas.

**Application: Simpson's rule.** For Simpson's rule, we have

$$I(f) = \frac{b-a}{6} \left( f(a) + 4f\left(\frac{1}{2}(a+b)\right) + f(b) \right).$$

You will prove on the next problem set that Simpson's rule is exact for cubic polynomials.<sup>§</sup> Thus, the Peano kernel theorem gives that

$$E(f) = \frac{1}{3!} \int_a^b f^{(4)}(t) K(t) dt,$$

where  $K(t) = E((x-t)_+^3)$ .

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<sup>§</sup>Again we emphasize that one needs to know  $n$ , the maximum degree polynomial for which the integration rule is exact, in advance; the Peano kernel theorem does not reveal  $n$  itself.

As with the analysis for the trapezoid rule, we begin by writing  $K(t)$  in a more explicit fashion:

$$\begin{aligned} K(t) &= E((x-t)_+^3) = \int_a^b (x-t)_+^3 dx - I((x-t)_+^3) \\ &= \frac{(x-t)^4}{4} \Big|_{x=t}^b - \frac{b-a}{6} \left( (a-t)_+^3 + 4\left(\frac{1}{2}(a+b)-t\right)_+^3 + (b-t)_+^3 \right) \\ &= \frac{(b-t)^4}{4} - \frac{b-a}{6} \left( 4\left(\frac{1}{2}(a+b)-t\right)_+^3 + (b-t)_+^3 \right), \end{aligned}$$

where we have used the fact that  $(a-t)_+ = 0$  for  $t \in [a, b]$ .

One can show that  $K(t)$  does not change sign for  $t \in [a, b]$ , and thus the mean value theorem for integrals guarantees the existence of some  $\xi \in [a, b]$  such that

$$E(f) = \frac{1}{3!} \int_a^b f^{(4)}(t)K(t) dt = \frac{1}{3!} f^{(4)}(\xi) \int_a^b K(t) dt.$$

Thus, we can integrate the ‘simplified’ formula for  $K(t)$  we derived above to get a clean expression for  $E(f)$ :

$$\begin{aligned} \int_a^b K(t) dt &= \int_a^b \frac{(b-t)^4}{4} dt - \frac{b-a}{6} \int_a^b \left( 4\left(\frac{1}{2}(a+b)-t\right)_+^3 + (b-t)_+^3 \right) dt \\ &= -\frac{(b-t)^5}{20} \Big|_{t=a}^b - \frac{b-a}{6} \left[ 4 \int_a^{(a+b)/2} \left(\frac{1}{2}(a+b)-t\right)^3 dt + \int_a^b (b-t)^3 dt \right] \\ &= -\frac{(b-t)^5}{20} \Big|_{t=a}^b - \frac{b-a}{6} \left[ \left(\frac{a-b}{2}\right)^4 - \frac{(b-a)^4}{4} \right] \\ &= -\frac{(b-a)^5}{480}. \end{aligned}$$

Inserting this into the expression for  $E(f)$ , we obtain

$$E(f) = -\frac{1}{90} \left(\frac{b-a}{2}\right)^5 f^{(4)}(\xi).$$

When performing such analysis yourself, it is a good idea to conduct the tedious calculations in a symbolic mathematics program, such as Maple or Mathematica, to help you avoid minor calculus/algebra mistakes.