Lecture 3: Projectors and Reflectors

1.1.2. Projectors.

Definition. A matrix $\mathbf{P} \in \mathbb{C}^{n\times n}$ is a *projector* provided $\mathbf{P}^2 = \mathbf{P}$. If $\mathbf{P}$ is also Hermitian, then $\mathbf{P}$ is an *orthogonal projector*. (Matrix powers imply repeated matrix multiplication: $\mathbf{P}^2 = \mathbf{P}\mathbf{P}$, etc.)

In this course, we shall be chiefly concerned with orthogonal projectors, which have appealing analytical and numerical properties.† The term 'orthogonal' refers to the nature of the projection: any vector orthogonal to the range of $\mathbf{P}$ is projected to the zero vector. Recall that the Fundamental Theorem of Linear Algebra ensures that $\mathrm{Ran}(\mathbf{P}) \perp \mathrm{Ker}(\mathbf{P}^*)$. When $\mathbf{P}$ is Hermitian, i.e., $\mathbf{P} = \mathbf{P}^*$, then $\mathrm{Ker}(\mathbf{P}^*) = \mathrm{Ker}(\mathbf{P})$, so $\mathrm{Ran}(\mathbf{P}) \perp \mathrm{Ker}(\mathbf{P})$.

Example. For any nonzero $\mathbf{v} \in \mathbb{C}^n$, the matrix

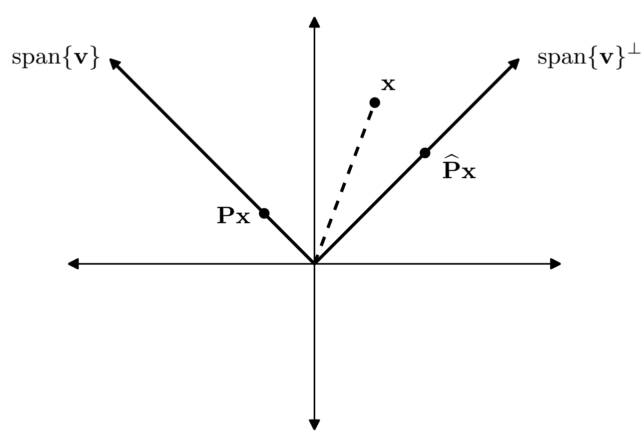
$$\mathbf{P} = \frac{\mathbf{v}\mathbf{v}^*}{\mathbf{v}^*\mathbf{v}}$$

is an orthogonal projector. Note that $\mathrm{Ran}(\mathbf{P}) = \mathrm{span}\{\mathbf{v}\}$, while $\mathrm{Ker}(\mathbf{P}) = \mathrm{span}\{\mathbf{v}\}^\perp = \{\mathbf{y} \in \mathbb{C}^n : \mathbf{y}^*\mathbf{v} = 0\}$. The matrix

$$\widehat{\mathbf{P}} = \mathbf{I} - \mathbf{P} = \mathbf{I} - \frac{\mathbf{v}\mathbf{v}^*}{\mathbf{v}^*\mathbf{v}}$$

is also an orthogonal projector, with $\mathrm{Ran}(\widehat{\mathbf{P}}) = \mathrm{span}\{\mathbf{v}\}^\perp$ and $\mathrm{Ker}(\widehat{\mathbf{P}}) = \mathrm{span}\{\mathbf{v}\}$.

The illustration below shows the effect of the projectors $\mathbf{P}$ and $\widehat{\mathbf{P}}$ in $\mathbb{R}^2$.



Example. For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ with $\mathbf{u}^*\mathbf{v} \neq 0$, the matrix

$$\mathbf{\Pi} = \frac{\mathbf{v}\mathbf{u}^*}{\mathbf{u}^*\mathbf{v}},$$

is a projector. When $\mathbf{u}$ and $\mathbf{v}$ are not collinear, $\mathbf{\Pi}$ is not Hermitian, and hence it is an *oblique* (not orthogonal) projector. For this example, $\mathrm{Ran}(\mathbf{\Pi}) = \mathrm{span}\{\mathbf{v}\}$ and $\mathrm{Ker}(\mathbf{\Pi}) = \mathrm{span}\{\mathbf{u}\}^\perp$. Can you replicate the sketch above, but now using this oblique projector?

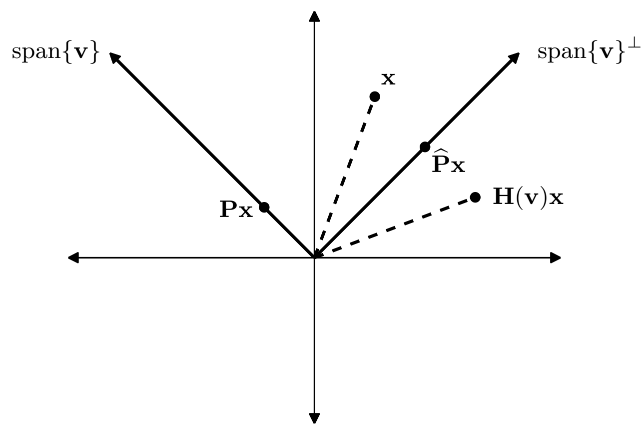
† Be sure to note the difference between *orthogonal projectors* and *orthogonal matrices*. The latter term refers to unitary matrices with real entries. Such matrices have full rank, and all columns are orthogonal and have norm 1. Orthogonal projectors have rank less than n (except in the trivial case of $\mathbf{P} = \mathbf{I}$), and the columns have norm less than or equal to one.

1.2. QR factorization.

The QR factorization is the first matrix decomposition we shall study. The goal is to write any $\mathbf{A} \in \mathbb{C}^{m\times n}$ in the form $\mathbf{A} = \mathbf{QR}$, where $\mathbf{Q} \in \mathbb{C}^{m\times m}$ is unitary and $\mathbf{R} \in \mathbb{C}^{m\times n}$ is *upper triangular*, i.e., $r_{jk} = 0$ if $j > k$. This decomposition is a workhorse: it will enable the efficient and stable solution of linear systems and least squares problems. Moreover, this factorization forms the cornerstone of the classic QR algorithm for eigenvalue computations.

1.2.1. Householder reflectors.

Householder reflectors are unitary matrices that are closely allied with the orthogonal projectors $\mathbf{P}$ and $\widehat{\mathbf{P}}$ discussed above. We wish to reflect a given vector $\mathbf{x} \in \mathbb{C}^n$ across an $n-1$ dimensional hyperplane. Perhaps this is most easily explained by amending the two-dimensional projector illustration given above.



We wish to reflect $\mathbf{x} \in \mathbb{C}^n$ over the $n-1$ dimensional subspace $\text{span}\{\mathbf{v}\}^\perp$. We will encode this operation in the matrix $\mathbf{H}(\mathbf{v}) \in \mathbb{C}^{n\times n}$, so that $\mathbf{H}(\mathbf{v})\mathbf{x} \in \mathbb{C}^n$ is the reflected vector.

How should $\mathbf{H}(\mathbf{v})$ be constructed? Look again at the illustration above. If we subtract $\mathbf{Px}$ from $\mathbf{x}$, (using simple head-to-tail vector subtraction) we get halfway to our goal – that is, we get $\widehat{\mathbf{P}}\mathbf{x} \in \text{span}\{\mathbf{v}\}^\perp$. To get the complete reflection across $\text{span}\{\mathbf{v}\}^\perp$, we simply subtract $\mathbf{Px}$ once more. In summary, this gives

$$\mathbf{H}(\mathbf{v})\mathbf{x} = \mathbf{x} - 2\,\mathbf{Px} = (\mathbf{I} - 2\mathbf{P})\mathbf{x} = \Big(\mathbf{I} - 2\,\frac{\mathbf{vv}^*}{\mathbf{v}^*\mathbf{v}}\Big)\mathbf{x}.$$

Definition. The matrix

$$\mathbf{H}(\mathbf{v}) = \mathbf{I} - 2\frac{\mathbf{vv}^*}{\mathbf{v}^*\mathbf{v}}$$

is called a *Householder reflector*.‡ For any $\mathbf{x} \in \mathbb{C}^n$, the vector $\mathbf{H}(\mathbf{v})\mathbf{x}$ is the reflection of $\mathbf{x}$ over the $n-1$ dimensional hyperplane $\text{span}\{\mathbf{v}\}^\perp$.

Notice in the above figure that $\mathbf{H}(\mathbf{v})\mathbf{x}$ has the same Euclidean length (i.e., 2-norm) as $\mathbf{x}$. This is no accident; it is a consequence of the fact that $\mathbf{H}(\mathbf{v})$ is a unitary matrix. You can verify this algebraically,

$$\mathbf{H}(\mathbf{v})^*\mathbf{H}(\mathbf{v}) = \Big(\mathbf{I} - 2\frac{\mathbf{vv}^*}{\mathbf{v}^*\mathbf{v}}\Big)\Big(\mathbf{I} - 2\frac{\mathbf{vv}^*}{\mathbf{v}^*\mathbf{v}}\Big) = \mathbf{I} - 2\frac{\mathbf{vv}^*}{\mathbf{v}^*\mathbf{v}} - 2\frac{\mathbf{vv}^*}{\mathbf{v}^*\mathbf{v}} + 4\frac{\mathbf{vv}^*\mathbf{vv}^*}{\mathbf{v}^*\mathbf{vv}^*\mathbf{v}} = \mathbf{I} - 4\frac{\mathbf{vv}^*}{\mathbf{v}^*\mathbf{v}} + 4\frac{\mathbf{vv}^*}{\mathbf{v}^*\mathbf{v}} = \mathbf{I}.$$

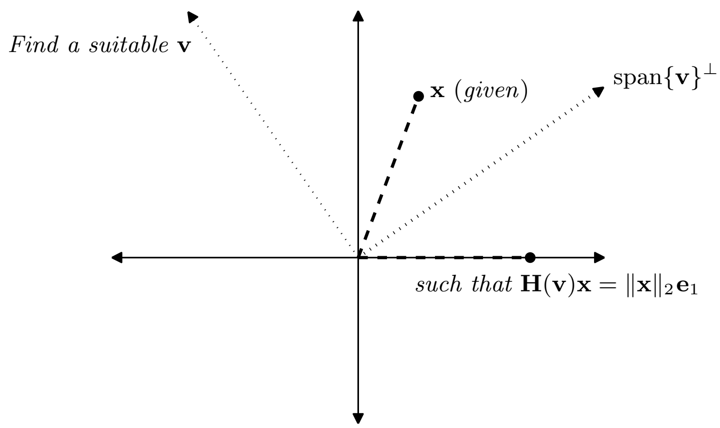
‡The reflectors are named for Alston Householder, who proposed this reflection and the QR factorization itself in the seminal 4-page paper "Unitary Triangularization of a Nonsymmetric Matrix," *J. ACM* 5 (1958) 339–342.

You can also appeal to geometric intuition: if you reflect a vector twice over the same hyperplane, you must get back to exactly where you started, so $\mathbf{H}(\mathbf{v})^2 = \mathbf{I}$. Since $\mathbf{H}(\mathbf{v})$ is Hermitian, this implies that $\mathbf{H}(\mathbf{v})^*\mathbf{H}(\mathbf{v}) = \mathbf{I}$, i.e., $\mathbf{H}(\mathbf{v})$ is unitary.

1.2.2. Using Householder reflectors to zero entries of a vector.

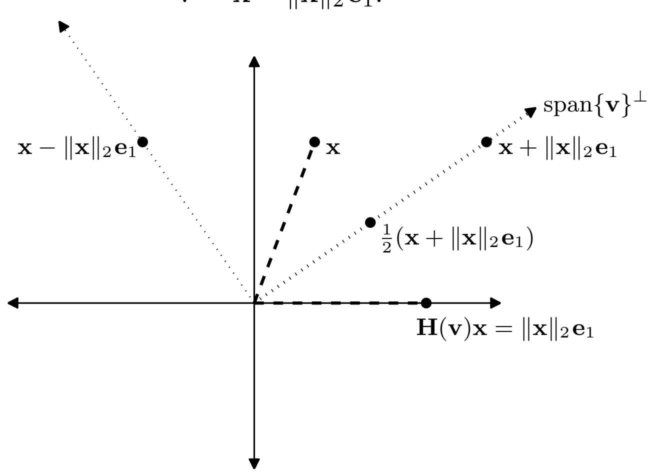
Householder reflectors are the fundamental tool needed for QR factorization. We shall see that the entire operation can be reduced to one sub-problem: Given a vector $\mathbf{x}$, find the vector $\mathbf{v}$ such that the reflector $\mathbf{H}(\mathbf{v})$ maps $\mathbf{x}$ to the vector $\|\mathbf{x}\|_2\mathbf{e}_1$, where $\mathbf{e}_1 = [1, 0, \ldots, 0]^T$; i.e.,

$$\mathbf{H}(\mathbf{v})\mathbf{x} = \begin{bmatrix} \|\mathbf{x}\|_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$



Here is one way to derive $\mathbf{v}$, illustrated in the figure below. Note that the midpoint $\frac{1}{2}(\mathbf{x} + \|\mathbf{x}\|_2\mathbf{e}_1)$ between $\mathbf{x}$ and $\|\mathbf{x}\|_2\mathbf{e}_1$ must lie on $\text{span}\{\mathbf{v}\}^\perp$, and so too must the collinear point $\mathbf{x} + \|\mathbf{x}\|_2\mathbf{e}_1$. To find $\mathbf{v}$, we simply need to find a vector that is both orthogonal to $\mathbf{x} + \|\mathbf{x}\|_2\mathbf{e}_1$ and in the plane spanned by $\mathbf{x}$ and $\mathbf{e}_1$.[§] For real $\mathbf{x}$, verify (by hand, and also by eye on the plot below) that a suitable choice for this orthogonal vector is

$$\mathbf{v} = \mathbf{x} - \|\mathbf{x}\|_2\mathbf{e}_1.$$



[§] If $\mathbf{v}$ is orthogonal to $\mathbf{x} + \|\mathbf{x}\|_2\mathbf{e}_1$ but not in the plane spanned by $\mathbf{x}$ and $\mathbf{e}_1$, then $\mathbf{x}$ will not generally be reflected to $\|\mathbf{x}\|_2\mathbf{e}_1$. To observe this, one has to look in three (or more) dimensions.

We shall not dwell on the case of complex $\mathbf{x}$, which requires a bit more care. (For completeness, note that we can set $\mathbf{v} = \mathbf{x} - e^{i\theta}\|\mathbf{x}\|\mathbf{e}_1$, giving $\mathbf{H}(\mathbf{v})\mathbf{x} = e^{i\theta}\|\mathbf{x}\|_2\mathbf{e}_1$, where $\theta = \arg(\mathbf{e}_1^*\mathbf{x})$.)

Example. We can verify the efficacy of the Householder reflector we have constructed with a simple MATLAB calculation.

```
>> x = [1;2;3;4;5]
x =
     1
     2
     3
     4
     5

>> norm(x)
ans =
    7.4162

>> v = x - norm(x)*[1;0;0;0;0]
v =
   -6.4162
    2.0000
    3.0000
    4.0000
    5.0000

>> Hv = eye(5)-2*v*v'/(v'*v)
Hv =
    0.1348    0.2697    0.4045    0.5394    0.6742
    0.2697    0.9159   -0.1261   -0.1681   -0.2102
    0.4045   -0.1261    0.8109   -0.2522   -0.3152
    0.5394   -0.1681   -0.2522    0.6638   -0.4203
    0.6742   -0.2102   -0.3152   -0.4203    0.4746

>> Hv*x
ans =
    7.4162
    0.0000
    0.0000
   -0.0000
    0.0000
```