Lecture 32: Absolute Stability

At this point, it may well seem that we have a complete theory for linear multistep methods. With an understanding of truncation error and zero stability, the convergence of any method can be easily understood. However, one further wrinkle remains. (You might have expected this: thus far the β_j coefficients have played no role in our stability analysis!) Up to this point, our convergence theory addresses the case where $h \to 0$. Methods differ significantly in how small h must be before one observes this convergent regime. For h too large, exponential errors that resemble those seen for zero-unstable methods can emerge for rather benign-looking problems—and for some ODEs and methods, the restriction imposed on h to avoid such behavior can be severe. To understand this problem, we need to consider how the numerical method behaves on a less trivial canonical model problem. (For an elaboration of many details described here, see Chapter 12 of Süli and Mayers.)

5.2.4. Absolute Stability.

Now consider the model problem $x'(t) = \lambda x(t)$, $x(0) = x_0$ for some fixed $\lambda \in \mathbb{C}$, which has the exact solution $x(t) = \mathrm{e}^{t\lambda}x_0$. In those cases where the real part of λ is negative (i.e., λ is in the open left half of the complex plane), we have $|x(t)| \to 0$ as $t \to \infty$. For a fixed step size $h > 0$, will a linear multistep method mimic this behavior? The explicit Euler method applied to this equation takes the form

$$\begin{aligned} x_{k+1} &= x_k + hf_k \\ &= x_k + h\lambda x_k \\ &= (1 + h\lambda)x_k. \end{aligned}$$

Hence, this recursion has the general solution

$$x_k = (1 + h\lambda)^k x_0.$$

Under what conditions will $x_k \to 0$? Clearly we need $|1 + h\lambda| < 1$; this condition is more easily interpreted by writing $|1 + h\lambda| = |-1 - h\lambda|$, where that latter expression is simply the distance of $h\lambda$ from -1 in the complex plane. Hence $|1 + h\lambda| < 1$ provided $h\lambda$ is located strictly in the interior of the disk of radius 1 in the complex plane, centered at -1. This is the *stability region* for the explicit Euler method, shown in the plot on the next page.

Now consider the backward (implicit) Euler method for this same model problem:

$$\begin{aligned} x_{k+1} &= x_k + hf_{k+1} \\ &= x_k + h\lambda x_{k+1}. \end{aligned}$$

Solve this equation for x_{k+1} to obtain

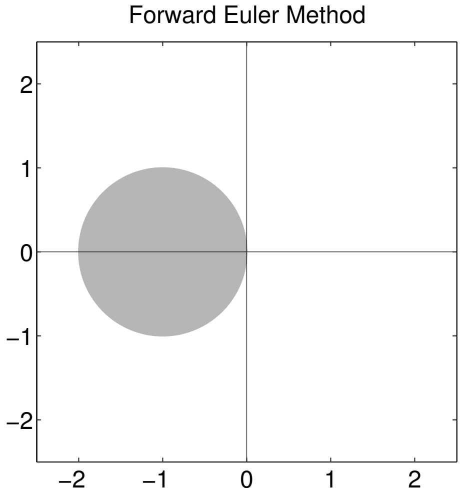
$$x_{k+1} = \frac{1}{1 - h\lambda} x_k,$$

from which it follows that

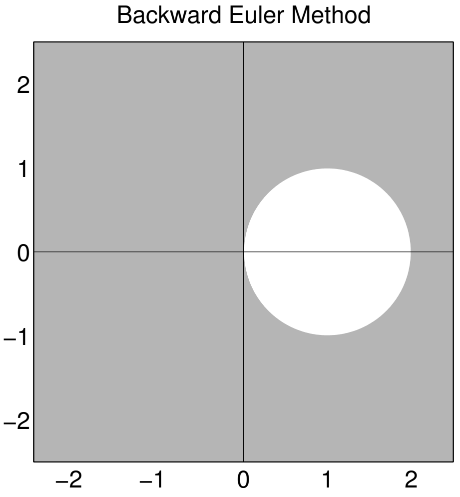
$$x_k = (1 - h\lambda)^{-k} x_0.$$

Thus $x_k \to 0$ provided $|1 - h\lambda| > 1$, i.e., $h\lambda$ must be *more than* a distance of 1 away from 1 in the complex plane. As illustrated in the plot on the next page, the backward Euler method has a much larger stability region than the explicit Euler method. In fact, the entire left half of the complex plane is contained in the stability region for the implicit method. Since $h > 0$, for any value of λ with negative real part, the backward Euler method will produce decaying solutions that qualitatively mimic the exact solution.

If $h\lambda$ falls within the stability region for a method, we say that the method is *absolutely stable* for that value of $h\lambda$. The stability regions for the explicit and backward Euler methods are shown below. The gray region shows values of λh in the complex plane for which the method is absolutely stable. (For the implicit method, this regions extend beyond the range of the plot.)


Forward Euler Method

$$x_{k+1} = x_k + hf_k$$


Backward Euler Method

$$x_{k+1} = x_k + hf_{k+1}$$

A general linear multistep method

$$\sum_{j=0}^{m} \alpha_j x_{k+j} = h \sum_{j=0}^{m} \beta_j f_{k+j}$$

applied to $x'(t) = \lambda x$, $x(0) = x_0$ reduces to

$$\sum_{j=0}^{m} \alpha_j x_{k+j} = h\lambda \sum_{j=0}^{m} \beta_j x_{k+j},$$

which can be rearranged as

$$\sum_{j=0}^{m} (\alpha_j - h\lambda\beta_j) x_{k+j}.$$

Note that this closely resembles the equation we analyzed when assessing the zero stability of linear multistep methods, except that now we have the $h\lambda\beta_j$ terms. Still this remains a linear recurrence relation, and just as before we can assume that it has solutions of the form $x_k = \gamma^k$ for constant γ. The values of $\gamma \in \mathbb{C}$ for which such x_k will be solutions to the recurrence are the roots of the *stability polynomial*

$$\sum_{j=0}^{m} (\alpha_j - h\lambda\beta_j) z^j,$$

which can be written as

$$\rho(z) - h\lambda\sigma(z) = 0,$$

where ρ is the characteristic polynomial,

$$\rho(z) = \sum_{j=0}^{m} \alpha_j z^j$$

and

$$\sigma(z) = \sum_{j=0}^{m} \beta_j z^j.$$

Thus for a fixed $h\lambda$, there will be m solutions of the form γ_j^k for the m roots $\gamma_1, \ldots, \gamma_m$ of the stability polynomial. If these roots are all distinct, then for any initial data $x_0, \ldots, x_{m-1}$ we can find constants $c_1, \ldots, c_m$ such that

$$x_k = \sum_{j=1}^{m} c_j \gamma_j^k.$$

For a given value $h\lambda$, we have $x_k \to 0$ provided that $|\gamma_j| < 1$ for all $j = 1, \ldots, m$. If that condition is met, we say that the linear multistep method is *absolutely stable* for that value of $h\lambda$.

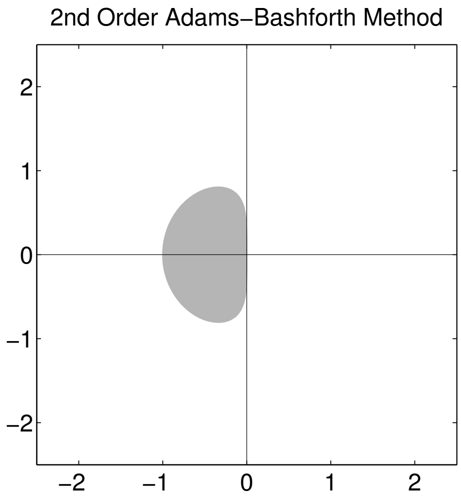
We seek linear multistep methods that share the following properties:

- high order truncation error;
- zero stability;
- absolute stability region that contains as much of the left half of the complex plane as possible.

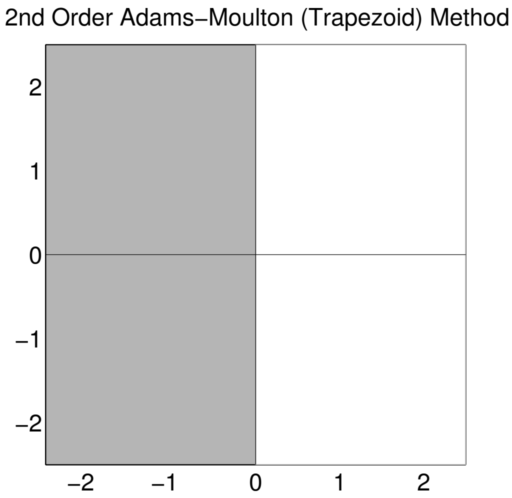
Those methods for which the stability region contains the entire left half plane are distinguished, as they will produce, *for any value of* h, exponentially decaying numerical solutions to linear problems that have exponentially decaying true solutions, i.e., when $\operatorname{Re}\lambda < 0$.

Definition. A linear multistep method is *A-stable* provided that its stability region contains the entire left half of the complex plane.

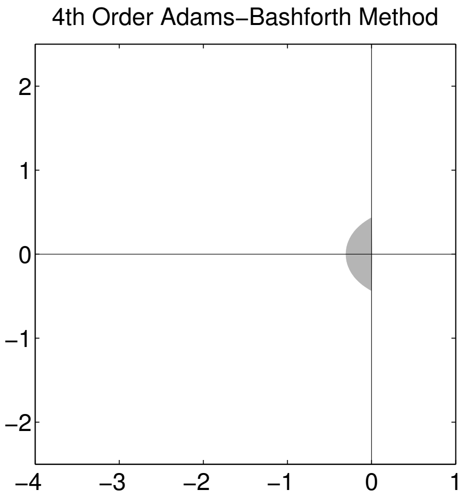
Next we show the stability regions for several different methods.


2nd Order Adams–Bashforth Method

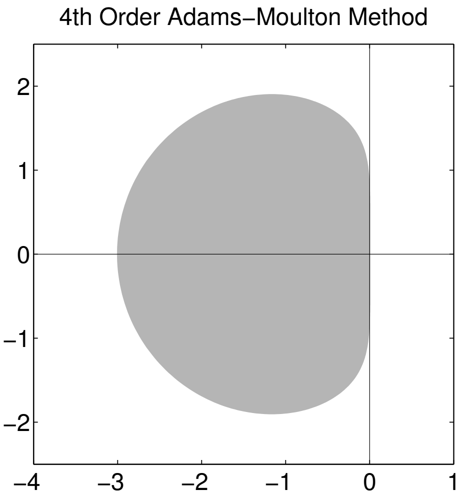
$$x_{k+2} - x_{k+1} = h(\tfrac{3}{2}f_{k+1} - \tfrac{1}{2}f_k)$$


2nd Order Adams–Moulton (Trapezoid) Method

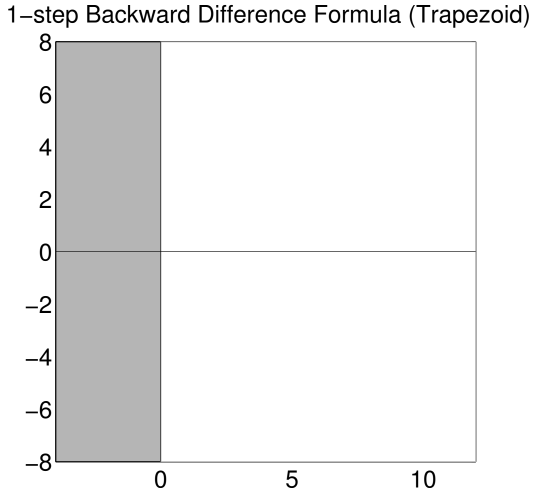
$$x_{k+1} - x_k = \tfrac{1}{2}h(f_k + f_{k+1})$$


4th Order Adams–Bashforth Method

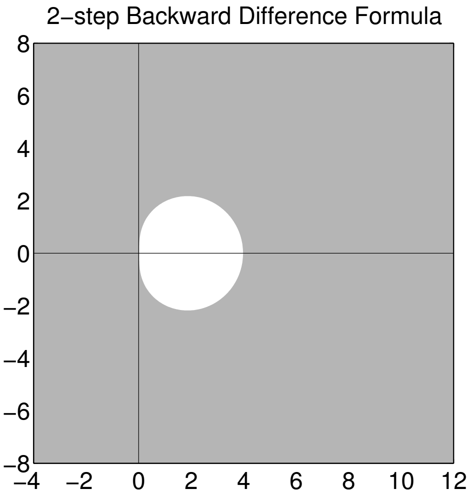
$$24x_{k+4} - 24x_{k+3} = h\big(55f_{k+3} - 59f_{k+2} + 37f_{k+1} - 9f_k\big)$$


4th Order Adams–Moulton Method

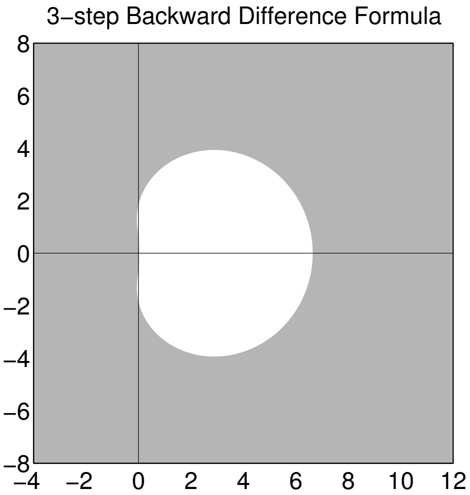
$$24x_{k+3} - 24x_{k+2} = h\big(9f_{k+3} + 19f_{k+2} - 5f_{k+1} + f_k\big)$$



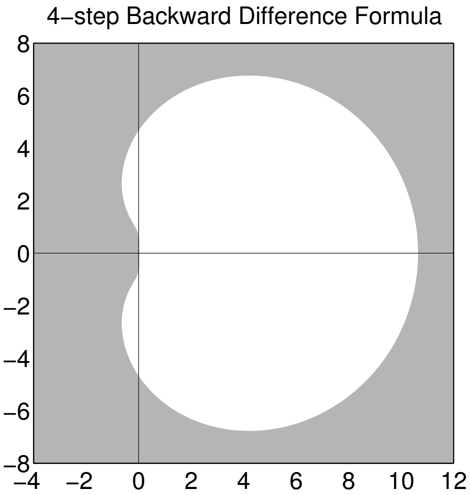
$$x_{k+1} - x_k = \tfrac{1}{2}h(f_k + f_{k+1})$$



$$3x_{k+2} - 4x_{k+1} + x_k = 2hf_{k+2}$$



$$11x_{k+3} - 18x_{k+2} + 9x_{k+1} - 2x_k = 6hf_{k+3}$$



$$25x_{k+4} - 48x_{k+3} + 36x_{k+2} - 16x_{k+1} + 3x_k = 12hf_{k+4}$$

How does one draw plots of the sort shown here? We take the second order Adams–Bashforth method

$$x_{k+2} - x_{k+1} = h(\tfrac{3}{2}f_{k+1} - \tfrac{1}{2}f_k)$$

as an example. Apply this rule to $x'(t) = f(t, x(t)) = \lambda x(t)$ to obtain

$$x_{k+2} - x_{k+1} = \lambda h(\tfrac{3}{2}x_{k+1} - \tfrac{1}{2}x_k),$$

with which we associate the stability polynomial

$$z^2 - (1 + \tfrac{3}{2}\lambda h)z + \tfrac{1}{2}\lambda h = 0.$$

Any point $\lambda h \in \mathbb{C}$ on the boundary of the stability region must be one for which the stability polynomial has a root z with $|z| = 1$. We can rearrange the stability polynomial to give

$$\lambda h = \frac{z^2 - z}{\frac{3}{2}z - 1}.$$

For general methods, this expression takes the form

$$\lambda h = \frac{\sum_{j=0}^{m} \alpha_j z^j}{\sum_{j=0}^{m} \beta_j z^j},$$

To determine the boundary of the stability region, we sample this formula for all $z \in \mathbb{C}$ with $|z| = 1$, i.e., we trace out the image for $z = \mathrm{e}^{\mathrm{i}\theta}$, $\theta \in [0, 2\pi)$. This curve will divide the complex plane into stable and unstable regions, which can be distinguished by testing the roots of the stability polynomial for λh within each of those regions.

We illustrate this process for the fourth order Adams–Bashforth scheme. The curve described in the last paragraph is shown in the plot below; it divides the complex plane into regions where the stability polynomial has an equal numbers of roots larger than 1 in magnitude. As denoted by the numbers on the plot: outside the curve there is one root larger than one; within the rightmost lobes of this curve, two roots are larger than one; within the leftmost region, no roots are larger than one in magnitude. The latter is the stable region, as shown in the plot several pages earlier.