CAAM 402/502 Spring 2013
Homework 2
Solutions

1. Problem XVII.1.1 in Lang. Let E be a vector space and let $v_1, ..., v_n \in E$ be a basis for E. Show that any linear map $\lambda : E \to F$ into the normed vector space F is continuous.

Proof. Let us denote the norm of F by $\|\cdot\|_F$. We also equip the finite dimensional vector space E with the norm $\|\cdot\|_\infty$ defined in terms of the basis $\{v_1, ..., v_n\}$ as follows. Given $x \in E$, let $x = \sum_{i=1}^n \alpha_i v_i$ where the scalars α_i are uniquely determined by x. Then $\|x\|_\infty = \max_i |\alpha_i|$. We brake the proof into 2 claims.

Claim 1: The linear map $\lambda : (E, \|\cdot\|_\infty) \to (F, \|\cdot\|_F)$ is continuous. Proof: In virtue of the result found in Lang page 455, we may equivalently prove that λ is bounded. Let $x \in E$ be arbitrary. There exist scalars $\{\alpha_1, ..., \alpha_n\}$ uniquely determined by x such that $x = \sum_{i=1}^n \alpha_i v_i$. Then

$$\|\lambda(x)\|_F = \left\| \sum_{i=1}^n \alpha_i \lambda(v_i) \right\|_F \le \sum_{i=1}^n |\alpha_i| \|\lambda(v_i)\|_F \le \left(\max_i |\alpha_i| \right) \sum_{i=1}^n \|\lambda(v_i)\|_F = C\|x\|_\infty$$

where $C = \sum_{i=1}^n \|\lambda(v_i)\|_F$ is clearly independent of x. Hence λ is bounded.

Claim 2: Let $\|\cdot\|_E$ be any norm on E. Then the linear map $\lambda : (E, \|\cdot\|_E) \to (F, \|\cdot\|_F)$ is continuous. Proof: Again we show boundedness. We use Theorem 4.3 (Lang page 145) on the equivalence of norms for finite-dimensional spaces. Then there exists a constant $K > 0$ such that

$$\|x\|_\infty \le K\|x\|_E, \quad \text{for all } x \in E$$

Then, from Claim 1, we have that,

$$\|\lambda(x)\|_F \le C\|x\|_\infty \le CK\|x\|_E, \quad \text{for all } x \in E$$

which makes λ a bounded map.

□

2. Let $f : [-1, 1] \to \mathbb{R}$ de defined as

$$f(x) = \begin{cases} 3x^2, & \text{if } x \in \mathbb{Q} \cap [-1, 1]; \\ 0, & \text{otherwise.} \end{cases}$$

Show that f is not differentiable at any point in $[-1, 1]$ except for $x = 0$.

Proof. Since differentiability implies continuity, then f cannot be differentiable at points where it is not continuous. Now we show that f is differentiable at $x = 0$ and that $f'(0) = 0$. Consider $h \neq 0$ and

$$0 \le \left| \frac{f(h) - f(0)}{h} \right| = \frac{f(h)}{|h|} \le \frac{3h^2}{|h|} = 3|h|.$$

So in the limit as $h \to 0$, we obtain that $f'(0) = 0$. □

3. Problem XVII.1.4 in Lang. Let E, F and G be normed vector spaces. A map

$$\lambda : E \times F \to G$$

is said to be *bilinear* if it satisfies the conditions,

$$\begin{aligned} \lambda(v, w_1 + w_2) &= \lambda(v, w_1) + \lambda(v, w_2) \\ \lambda(v_1 + v_2, w) &= \lambda(v_1, w) + \lambda(v_2, w) \\ \lambda(cv, w) &= c\lambda(v, w) = \lambda(v, cw) \end{aligned}$$

for all v, $v_i \in E$, w, $w_i \in F$ and $c \in \mathbb{R}$

(a) Show that a bilinear map λ is continuous if and only if there exists ($C > 0$) such that for all $(v, w) \in E \times F$ we have

$$|\lambda(v, w)| \leq C|v||w|$$

Proof. "$\Rightarrow$" , λ is continuous $\Rightarrow$ λ is continuous at $(0, 0)$. This means that there exist $\delta_1, \delta_2 > 0$ such that

$$|v_i| \leq \delta_1, |w_i| \leq \delta_2 \Rightarrow |\lambda(v_i, w_i) - \lambda(0, 0)| \leq 1$$

We have $\lambda(0, 0) = 0$ from third property of bilinear map. Hence

$$|v_i| \leq \delta_1, |w_i| \leq \delta_2 \Rightarrow |\lambda(v_i, w_i)| \leq 1$$

Consider for any $v \in E$, $w \in F$, $v_i = \dfrac{v}{|v|}\delta_1$ and $w_i = \dfrac{w}{|w|}\delta_2$. Clearly $|v_i| = \delta_1$ and $|w_i| = \delta_2$. Therefore we have,

$$\lambda\left(\frac{v}{|v|}\delta_1, \frac{w}{|w|}\delta_2\right) \leq 1 \Rightarrow \frac{\delta_1}{|v|}\frac{\delta_2}{|w|}|\lambda(v, w)| \leq 1 \Rightarrow |\lambda(v, w)| \leq \frac{1}{\delta_1\delta_2}|v||w| \Rightarrow |\lambda(v, w)| \leq C|v||w|$$

with $C = \dfrac{1}{\delta_1\delta_2}$.

"$\Leftarrow$" Assume that there exists $C > 0$ such that $|\lambda(v, w)| \leq C|v||w|$ for all $(v, w) \in E \times F$. For any $\epsilon > 0$ and $(v_i, w_i) \in E \times F$, pick $\delta = \min\left(1, \dfrac{\epsilon}{C(1 + |v_i| + |w_i|)}\right)$

$$\begin{aligned} |\lambda(v, w) - \lambda(v_i, w_i)| &= |\lambda(v - v_i, w_i) + \lambda(v - v_i, w - w_i) + \lambda(v_i, w - w_i)| \\ &\leq |\lambda(v - v_i, w_i)| + |\lambda(v - v_i, w - w_i)| + |\lambda(v_i, w - w_i)| \\ &\leq C|v - v_i||w_i| + C|v - v_i||w - w_i| + C|v_i||w - w_i| \\ &< C\delta|w_i| + C\delta^2 + C\delta|v_i| \qquad \text{for } |v - v_i| < \delta, |w - w_i| < \delta \\ &< \epsilon \end{aligned}$$

Hence λ is continuous at every $(v_i, w_i) \in E \times F$ □

(b) Let $v \in E$ be fixed. Show that if λ is continuous, then the map $\lambda_v : F \to G$ given by $w \to \lambda(v, w)$ is a continous linear map.

Proof. Let $\alpha, \beta \in \mathbb{R}$ and $w_1, w_2 \in F$.

$\lambda_v(\alpha w_1 + \beta w_2) = \lambda(v, \alpha w_1 + \beta w_2) = \lambda(v, \alpha w_1) + \lambda(v, \beta w_2) = \alpha\lambda(v, w_1) + \beta\lambda(v, w_2) = \alpha\lambda_v(w_1) + \beta\lambda_v(w_2)$

Hence λ_v is linear.

$$|\lambda_v(w)| = |\lambda(v, w)| \leq C|v||w| = C_v|w| \Rightarrow |\lambda_v(w)| \leq C_v|w| \qquad C_v = C|v|$$

Hence λ_v is bounded and linear. Hence λ_v is a continuous linear map. □

4. Let S be an open and connected set in $\mathbb{R}^n$, and consider the differentiable function $f : S \to \mathbb{R}^m$ with derivative $f'(c) = 0$ (the linear map zero), for all $c \in S$. Prove that f is constant. **Hint:** Use the result of Problem 1 below, which says that S is polygonally connected.

Proof. Fix $a \in S$ and let $x \in S$. Since S is open and connected, it is polygonally connected (Problem 1 for 502). Thus, there exist points $x_0 = a, x_1, \cdots, x_k = x$ such that $L(x_{j-1}, x_j)$ (line segment between two points) is in S for all $j = 1, \cdots, k$.
By the fundamental theorem of calculus we have that

$$f(x_j) - f(x_{j-1}) = \int_0^1 f'(tx_j + (1-t)x_{j-1})dt\ (x_j - x_{j-1}) = 0$$

for $j = 1, \cdots, k$. Summing over j we get

$$0 = \sum_{j=1}^{k} (f(x_j) - f(x_{j-1})) = f(x) - f(a).$$

Hence, $f(x) = f(a)$ for all $x \in S$, that is, f is constant. $\square$

ADDITIONAL PROBLEMS FOR CAAM 502

1. Let S be an open and connected set in $\mathbb{R}^n$. Prove that S is polygonally connected, ie., any two points x and y in S can be joined by a polygonal line.

Proof. Since S is connected, then the only set two sets in S that are both open and closed (with respect to S) are the whole set S and the empty set . Assume that S is not empty and let $a \in S$. Let $C \subset S$ denote the set of all the points in S that are polygonally connected to a. This set C is clearly nonempty since a is polygonally to itself. Now we claim that C is both open and closed with respect to S.

We proceed to prove openness first. Let $x \in C \subset S$. Since S is open in $\mathbb{R}^n$, then there exists $r > 0$ such that $B_r(x) \subset S$. Let $y \in B_r(x)$ be arbitrary. Notice that the line segment from x to y stays within $B_r(x) \subset S$ (by convexity). Now since $x \in C$ there is a polygonal line in S from a to x. We simply join this line with the line segment from x to y to obtain the desired polygonal line from a to y. Hence we have that $B_r(x) \subset C$ which implies that C is open in S.

Finally we proceed to show that C is closed with respect to S or equivalent that the complement $S \setminus C$ is open. Take $x \in S \setminus C$ arbitrary (no polygonall line from a to x). Since S is open in $\mathbb{R}^n$, then there exists $r > 0$ such that $B_r(x) \in S$. No point y in the ball $B_r(x)$ can belong to C, otherwise (like before) we could join a line segment to connected to the center of the ball which is x, and this cannot happen since $x \notin C$. Hence $B_r(x) \subset S \setminus C$ which makes $S \setminus C$ open in S.

Since a was chosen arbitrarily in S, then we conclude that S is polygonally connected. □

2. Let E and F be normed vector spaces and denote by $L^2(E, F)$ the space of continuous bilinear maps of $E \times E \to F$. Let $\lambda \in L^2(E, F)$ and define $\|\lambda\|$ to be the greatest lower bound of all numbers $C > 0$ such that

$$\|\lambda(v_1, v_2)\|_F \leq C\|v_1\|_E\|v_2\|_E \qquad \forall v_1, v_2 \in E$$

Show that $\|\lambda\|$ is a norm in $\lambda \in L^2(E, F)$ and verify that its definition is equivalent to

$$\|\lambda\| = \sup_{v_1, v_2 \neq 0} \frac{\|\lambda(v_1, v_2)\|_F}{\|v_1\|_E\|v_2\|_E}$$

Proof. First we verify the equivalence of definitions. This follows from the fact that the greatest lower bound of all C's that satisfy the above inequality coincides with the least upper bound of all C's that do not satisfy the same inequality. Now we check the axioms of a norm.

N1. $\|\lambda\| \geq 0$ and $\|\lambda\| = 0$ iff $\lambda = 0$
$\|\lambda\|$ is the greatest lower bound of all C's that satisfy the above inequality. Hence

$$\frac{\|\lambda(v_1, v_2)\|_F}{\|v_1\|_E\|v_2\|_E} \leq \|\lambda\|$$

$\|\cdot\|_F$ is a norm, hence $\|\lambda(v_1, v_2)\|_F \geq 0 \Rightarrow \|\lambda\| \geq 0$.
If $\|\lambda\| = 0 \Rightarrow \|\lambda(v_1, v_2)\|_F = 0$, $\forall v_1, v_2 \in E \Rightarrow \lambda(v_1, v_2) = 0$, $\forall v_1, v_2 \in E$. Hence $\lambda = 0$

N2. For any $\alpha \in \mathbb{R}$ $\|\alpha\lambda\| = |\alpha|\|\lambda\|$
For any $C > 0$ satisfying the given inequality, we have

$$\|\alpha\lambda(v_1, v_2)\|_F = |\alpha|\|\lambda(v_1, v_2)\|_F \leq |\alpha|C\|v_1\|_E\|v_2\|_E$$

If $\|\lambda\|$ is the greatest lower bound of all possible numbers C, $|\alpha|\|\lambda\|$ is the greatest lower bound of all possible numbers $|\alpha|C$. Hence $\|\alpha\lambda\| = |\alpha|\|\lambda\|$

N3. $\|\lambda_1 + \lambda_2\| \leq \|\lambda_1\| + \|\lambda_2\|$ for any $\lambda_1, \lambda_2 \in L^2(E, F)$

We have

$$\|\lambda_1(v_1, v_2)\|_F \leq \|\lambda_1\| \|v_1\|_E \|v_2\|_E$$

and

$$\|\lambda_2(v_1, v_2)\|_F \leq \|\lambda_2\| \|v_1\|_E \|v_2\|_E$$

by adding the above two inequalities,

$$\|\lambda_1(v_1, v_2)\|_F + \|\lambda_2(v_1, v_2)\|_F \leq \|\lambda_1\| \|v_1\|_E \|v_2\|_E + \|\lambda_2\| \|v_1\|_E \|v_2\|_E = (\|\lambda_1\| + \|\lambda_2\|) \|v_1\|_E \|v_2\|_E$$

From the norm property of $\|\cdot\|_F$ and the above inequality,

$$\|(\lambda_1 + \lambda_2)(v_1, v_2)\|_F \leq \|\lambda_1(v_1, v_2)\|_F + \|\lambda_2(v_1, v_2)\|_F \leq (\|\lambda_1\| + \|\lambda_2\|) \|v_1\|_E \|v_2\|_E$$

From the definition, $\|\lambda_1 + \lambda_2\| \leq C = \|\lambda_1\| + \|\lambda_2\|$.

$\square$

3. Let E and F be normed vector spaces and $\lambda \in L(E, L(E, F))$. Let also $f_\lambda \in L^2(E, F)$ be the continuous bilinear map defined by

$$f_\lambda(v_1, v_2) = [\lambda(v_1)](v_2) \qquad \forall v_1, v_2 \in E$$

Prove that $\|f_\lambda\| = \|\lambda\|$

Proof.

$$\|\lambda\| = \sup_{v_1 \neq 0} \frac{\|\lambda(v_1)\|_{L(E,F)}}{\|v_1\|_E} = \sup_{v_1 \neq 0} \frac{\sup_{v_2 \neq 0} \frac{\|[\lambda(v_1)](v_2)\|_F}{\|v_2\|_E}}{\|v_1\|_E} = \sup_{v_1, v_2 \neq 0} \frac{\|[\lambda(v_1)](v_2)\|_F}{\|v_2\|_E \|v_1\|_E} = \|f_\lambda\|$$

$\square$