Last time:

\[ \sum -\frac{\partial}{\partial x} \left( k(x) \frac{d\psi}{dx} \right) + c(x) \psi = f(x) \quad c(x) \geq 0. \]

Multiply by test function \( v \) and integrate gives

\[ a(u, v) = \int_a^b \left( k(x) u'(x) v'(x) + c(x) u(x) v(x) \right) dx = \int_a^b f(x) v(x) dx \]

where \( v(x) \) is chosen so \( v(a) = v(b) = 0 \).

\( c, f \in L^2(\Omega) \), \( k \in L^\infty(\Omega) \), \( u, v \in H^1_0(\Omega) \).

This is the variational problem.

The finite element method.

We choose our test function \( v_j \) to approximate the solution in \( V_h = \sum \text{VEC}(0, l) : v \mid_{\Omega_i} = 0 \)

Note \( V_h \) is finite dimensional.

Let \( \{ \phi_i \}_{i=1}^N \) denote a basis of \( V_h \).

Then our approximate solution is going to be given by

\[ u_h(x) = \sum_{j=1}^N a_j \phi_j(x). \]

Now our goal is to find the coefficients \( a_j \).

Plugging this into the variational problem with test function \( \phi_i \), we get

\[ a(\sum_{j=1}^N a_j \phi_j(x), \phi_i(x)) = \int_a^b f(x) \phi_i(x) dx \]

\[ \Rightarrow \sum_{j=1}^N a_j A(\phi_j(x), \phi_i(x)) = \int_a^b f(x) \phi_i(x) dx \]

Plugging this into the variational problem with test function \( \phi_i \), we get

\[ A \vec{a} = \vec{f} \]

Linear system.
often matrix is written as \( A = \hat{A} + M \).

Where \( \hat{A} \) is the stiffness matrix s.t. \( \hat{A}_{ij} = \int k(x) \phi_j'(x) \phi_i'(x) \, dx \),

\( M \) is the mass matrix. \( M_{ij} = \int \phi_j(x) \phi_i(x) \, dx \).

**Common Choice of basis** (piecewise linear): *hat functions*.

\[
\phi_i(x) = \begin{cases} 
0 & \text{if } 0 \leq x \leq x_{i-1} \\
\frac{(x-x_{i-1})}{h_i} & \text{if } x_{i-1} \leq x \leq x_i \quad (h_i = x_i - x_{i-1}) \\
\frac{(x-x_i)}{h_i} & \text{if } x_i \leq x \leq x_{i+1} \\
0 & \text{if } x_{i+1} \leq x \leq 1 
\end{cases}
\]

**Thm:** (1) The set of hat functions \( \sum \phi_i(x) \delta_{ij} \), are linearly independent.

**Proof:** Suppose \( \sum_{j=1}^{N} \alpha_j \phi_j(x) = 0 \) for some \( \alpha_j \in \mathbb{R} \).

WTS: that the only way this can be true is if \( \alpha_j = 0 \) \( \forall j \).

Evaluate (1) at a pt \( x_p \). Note \( \phi_j(x_i) = \delta_{ij} \)

\[ \Rightarrow \alpha_i = 0. \] This is true \( \forall i = 1, \ldots, N \).
(2) \[ \sum_{j=1}^{J} \phi_j \] span \( V_h \) (linear)

**Proof:** let \( v \in V_h \), let \( \widetilde{v(x)} = \sum_{j=0}^{N} v(x_j) \phi_j(x) \cdot 6V_h \).

**Goal:** show that \( \widetilde{v} = v \) in \( V_h \).

Let \( I = [x_k, x_{k+1}] \subset [0,1] \). where \( x_k, x_{k+1} \) are partition pts. then.

\[
\widetilde{v(x_{k+1})} = \sum_{j=0}^{N} v(x_j) \phi_j(x_{k+1}) = v(x_{k+1}).
\]

\( \Rightarrow \widetilde{v} \) agree ptwise on \( I \). since \( v \) are linear on \( I \). & polynomial interpolation is unique. \( \widetilde{v} = v \) on \( I \). \( \Rightarrow \widetilde{v} = v \) in \( V_h \).

**How Do we know the linear system \( A \bar{u} = \bar{F} \) is invertible?**

We know the bilinear form is symmetric, bounded. & elliptic in \( H_0^1(\Omega) \). since \( V_h \subseteq H_0^1(\Omega) \). the same holds of \( H_0^1(\Omega) \).

Since we know 3 unique solution.
Existence & Uniqueness.

Since this is a linear problem, Uniqueness \implies Existence.

Suppose we have two solutions \( U^1_n, U^2_n \).

Let \( W_n = U^1_n - U^2_n \). \( \text{Goal: show } W_n = 0. \)

We know by linearity
\[
 a(W_n, v) = a(U^1_n, v) - a(U^2_n, v) = l(v) - l(v) = 0 \quad \forall v \in V_n.
\]

Pick \( v = W_n \in V_n \).
\[
 a(W_n, W_n) = 0. \implies 0 = \int_0^1 k(x) (W_n)^2 \, dx + \int_0^1 c(x) W_n^2 \, dx \geq 0.
\]

\(
\implies \int_0^1 k(x) (W_n)^2 \, dx = 0.
\)

\( \implies W_n = 0 \implies W_n = \text{constant.} \)

We know \( W_n(0) = 0 \implies W_n = 0. \)

Since \( W_n \in C(0, 1). \)

A priori Error estimates:

Def.: \( \|v\|_\varepsilon = (a(v, v))^{\frac{1}{2}} \) is called the energy norm.

4. orthogonality eqn. \( \forall v \in V_n, a(u - u_n, v) = 0. \)

\( \forall v \in V_n, a(u - u_n, v) = 0. \)

Since \( a(u, v) = l(v) \)

\( \implies a(u_n, v) = l(v) \)
Lemma \( \forall v \in V_h \quad \| u - u_h \|_{\varepsilon} \leq \| u - v \|_{\varepsilon}. \)

**Proof:**
\[
\| u - u_h \|_{\varepsilon}^2 = a(u - u_h, u - u_h) \quad \text{by orthogonality}
\]
\[
= a(u - u_h, u - u) + a(u - u_h, u - v)
\]
\[
= a(u - u_h, u - v)
\]
\[
= \int_0^1 k(u - u_h) (u - v) \, dx + \int_0^1 \lambda (u - u_h)(u - v) \, dx.
\]

\[1 \leq \left( \int_0^1 k(u - u_h)^2 \right)^{\frac{1}{2}} \left( \int_0^1 k(u - v)^2 \right)^{\frac{1}{2}}.
\]

\[2 = \int_0^1 (u - u_h)^2 \left( \int_0^1 (u - v)^2 \right)^{\frac{1}{2}}.
\]

\[\Rightarrow \quad \| u - u_h \|_{\varepsilon}^2 \leq \| u - v \|_{\varepsilon} \| u - v \|_{\varepsilon} \]

Taking \( \inf \) over all \( v \in V_h \) and noting \( u \in V_h \),

\[\| u - u_h \|_{\varepsilon} = \inf_{v \in V_h} \| u - v \|_{\varepsilon} \]

Using interpolation error bounds, we find,

\[\inf_{v \in V_h} \| u - v \|_{\varepsilon} \leq C h^{1/2} \quad \text{Order 2 approximation}\]

i.e., piecewise linear \( \approx \) 1st order.

Weaker norm \( \| u - u_h \|_{L^2(\Omega)} \leq C h^{1/2} \).