What is a PDE?

A partial differential equation is an equation relating an unknown function $u(x_1, \ldots, x_n)$, the independent variables $x_1, \ldots, x_n$ and a finite number of partial derivatives of $u$, i.e.,

$$F(x_1, x_2, \ldots, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial^k u}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}) = 0$$

The order of a partial differential equation is that of the derivative of the highest order.

A partial differential equation is called quasi-linear if it is linear in all the highest order derivatives of the unknown function, e.g.,

$$\begin{align*}
Ex. \quad & A(x, y, u, u_x, u_y) \frac{d^2 u}{d x^2} + B(x, y, u, u_x, u_y) \frac{d^2 u}{d x d y} \\
& + C(x, y, u, u_x, u_y) \frac{d^2 u}{d y^2} + F(x, y, u, u_x, u_y) = 0.
\end{align*}$$

2nd order quasi-linear.

A PDE is called linear if it is linear in the unknown function and its partial derivatives. Ex.

$$\begin{align*}
Ex. \quad & A(x, y) \frac{d^2 u}{d x^2} + B(x, y) \frac{d^2 u}{d x d y} + C(x, y) \frac{d^2 u}{d y^2} \\
& + D(x, y) \frac{d u}{d x} + E(x, y) \frac{d u}{d y} + G(x, y) u(x, y) = F(x, y)
\end{align*}$$

2nd order linear PDE.
Poster Examples

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial t^2} \quad \text{Wave eqn}
\]

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{Laplace eqn}
\]

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} \quad \text{Heat eqn}
\]

Classification of 2nd order eqns.

Behavior is dictated by the high order terms. Consider

\[
al(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} = f(x,y, u, u_x, u_y)
\]

Where \(a(x,y), b(x,y), c(x,y)\) are twice continuously differentiable functions of \(xy\).

Goal: rewrite the problem in terms of something that resembles a polynomial.

The problems above, \(\Box\).

To do this, we use a change of variables.

\[
\alpha = \Phi(x, y) \quad \beta = \Psi(x, y).
\]

Since we want the transformed problem to be equivalent to the original eqn., we need the Jacobian to be non-zero.

\[
\frac{\partial (\Phi, \Psi)}{\partial (x, y)} = \begin{vmatrix} \Phi_x & \Phi_y \\ \Psi_x & \Psi_y \end{vmatrix} \neq 0.
\]
by the inverse function Theorem, this guarantees
\[ F \left( V, W \right) = \chi \Rightarrow x = V(\alpha, \beta), y = W(\alpha, \beta). \]

\[ \Rightarrow u \left( V(\alpha, \beta), W(\alpha, \beta) \right) = u(\alpha, \beta). \]

Now rewrite derivatives in terms of new variables.

\[ u_x = \beta \Phi_x + \alpha \Phi_y \]
\[ u_y = \alpha \Phi_x + \beta \Phi_y \]
\[ u_{xx} = \varepsilon \text{ etc.} \]

\[ \text{end up with transformed eqn.} \]

\[ A(x, \beta) u_{xx} + 2B(x, \beta) u_{xy} + C(x, \beta) u_{yy} = F(\alpha, \beta, u, u_x, u_y). \]

Where

\[ A = \alpha \psi_x^2 + 2b \psi_x \psi_y + c \psi_y^2 \]
\[ B = \alpha \psi_x^2 + 2b \psi_x \psi_y + c \psi_y^2 \]
\[ C = \alpha \psi_x \psi_y + b (\psi_x \psi_y + \psi_y \psi_x) + c \psi_y \psi_y \]

Assume either \( a \) or \( c \) \( \neq 0 \). \( \text{wlog (symmetry)} \)

Suppose \( a \neq 0 \).

Note, we can achieve \( A = C = 0 \) if both \( \psi_x \) \& \( \psi_y \)
Satisfy

\[ \alpha \psi_x^2 + 2b \psi_x \psi_y + c \psi_y^2 = 0. \]
\[ a (\psi_x - \frac{b}{\sqrt{\alpha}}) \left( \psi_x + \frac{b}{\sqrt{\alpha}} \right) \psi_y^2 = 0. \]
\[ t_1 = \psi_x \psi_y \]
\[ \text{Quadratic form} \ a t_1^2 + 2b t_1 t_2 + c t_2^2 = 0. \]

2) \( \text{HDI PDE: Solve via method of characteristics.} \)
Solutions behave like the quadratic

(2)
if \( b^2 - ac > 0 \) is the region of interest (wrt \( x,y \))
the PDE is called **hyperbolic**

- \( b^2 - ac < 0 \) is the region of interest
the PDE is called **elliptic**

- \( b^2 - ac = 0 \) is region of interest
it is called **Parabolic**

Ex. wave eqn. \( \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \) hyperbolic.

Laplace \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \) elliptic

Heat \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \) parabolic

**Generalization** (2nd order)
A PDE is of elliptic type if the problem can be transformed into the sum of \( n \) squares all the same sign.

- A PDE is of hyperbolic type if the problem can be reduced to the sum of \( n \) squares \( n-1 \) are of the same sign

- A PDE is of parabolic type if it can be reduced by a linear combination of transformations to the sum of fewer than \( n \) squares all not necessarily the same sign.

- ultra hyperbolic. sum of \( n \geq 4 \) squares with
more than 1 coefficient of either sign.