(1) (10 points) Show that the centered finite difference approximation $D^2 u(x)$ approximates $u''(x)$ with an error $O(h^2)$. Recall that 
\[
D^2 u(x) = \frac{u(x - h) - 2u(x) + u(x + h)}{h^2}.
\]

(2) (20 points) Let $h = \frac{1}{N+1}$. Define the tridiagonal matrix $A = (a_{ij})$ such that the only nonzero entries are
\[
a_{ii} = \frac{2}{h^2}, \quad 1 \leq i \leq N
\]
\[
a_{i,i+1} = -\frac{1}{h^2}, \quad 1 \leq i \leq N - 1
\]
\[
a_{i-1,i} = -\frac{1}{h^2}, \quad 2 \leq i \leq N
\]
Prove that the eigenpairs $(\lambda_k, u_k)$ of $A$ are 
\[
\lambda_k = \frac{2}{h^2} (1 - \cos(k\pi h))
\]
\[
u_k = \begin{bmatrix}
\sin(k\pi h) \\
\sin(2k\pi h) \\
\vdots \\
\sin((N-1)k\pi h) \\
\sin(Nk\pi h)
\end{bmatrix}
\]
for $1 \leq k \leq N$.

*Hint:* you need to check that $Au_k = \lambda_k u_k$.

(3) (30 points)
(a) Use the method of undetermined coefficients to set up the $5 \times 5$ Vandermonde system that would determine a fourth-order accurate finite difference approximation to $u(x)$ based on 5 equally spaced points,
\[
u''(x) = c_{-2} u(x - 2h) + c_{-1} u(x - h) + c_0 u(x) + c_1 u(x + h) + c_2 u(x + 2h) + O(h^4).
\]

*Hint:* See the CAAM 452 notes.

(b) Compute the coefficients using the matlab code `fdstencil.m` available from the website, and check that they satisfy the system you determined in part (a).

(c) Test this finite difference formula to approximate $u''(1)$ for $u(x) = \cos(3x)$ with values of $h$ from the array `hvals = logspace(-1, -4, 13)`. Make a table of the error vs. $h$ for several values of $h$ and compare against the predicted error from the leading term of the expression printed by `fdstencil`. Also produce a log-log plot of the absolute value of the error vs. $h$. Describe your results.

(4) (40 points) **This problem is pledged!** You may not discuss this with anyone but your instructor. You may not consult any source other than approved textbooks, CAAM 452 lecture notes or your in-class notes to help you with the problem.
(a) Implement the finite difference method of second order for solving
\[ u''(x) + cu(x) = f(x), \quad 0 < x < 1 \]
\[ u(0) = \alpha, \]
\[ u(1) = \beta \]

The parameter \( c \) is a nonnegative real number.

(b) Verify your code on the following three examples

1. The exact solution is \( u(x) = 1 - x \) and \( c = 0 \).
2. The exact solution is \( u(x) = x \) and \( c = 1 \).
3. The exact solution is \( u(x) = xe^{-x} \) and \( c = 1 \).

Consider a uniform partition of the interval \((0, 1)\) with \( M \) intervals. Make a table of the max-norm error vs. \( h = 1/M \) for \( M = 10, 20, 40, 80 \). Obtain the numerical convergence rate.

Challenge problem

Consider the boundary value problem
\[ -d \left( a(x) \frac{du}{dx} \right) = f(x), \quad u(0) = u(1) = 0 \]
where \( a(x) > \delta \geq 0 \) is a bounded differentiable function in \([0, 1]\).

(a) Using finite differences and an equally spaced grid in \([0, 1]\), \( x_l = hl, \quad l = 1, \ldots, n - 1 \) discretize the ODE to obtain a linear system of equations, with a tridiagonal matrix, yielding an \( O(h^2) \) approximation of the ODE. Provide a derivation and write down the resulting linear system.

(b) Utilizing all the information provided, find the smallest disc in \( \mathbb{C} \) that is guaranteed to contain all the eigenvalues of the linear system constructed in part (a).