CAAM 536
Homework 2
Due Feb 17

(1) (30 points) The matlab script poisson.m solves the Poisson problem on the unit square $(0, 1)^2$ with a grid size $\Delta x = \Delta y = h = 1/(N + 1)$, using the 5-point stencil. It is set up to solve a test problem for which the exact solution is $u(x, y) = \exp(x + y/2)$, using Dirichlet boundary conditions and the right hand side $f(x, y) = 1.25\exp(x + y/2)$.

(a) Test this script by performing a grid refinement study to verify that it is second order accurate: consider $h = 1/10, 1/20, 1/40, 1/80$.

(b) Modify the script so that it works on a rectangular domain $[a_x, b_x] \times [a_y, b_y]$, but still with $\Delta x = \Delta y = h$. Test your modified script on the domain $(0, 1) \times (1, 3)$ and use the values $h = 1/10, 1/20, 1/40, 1/80$. Give the errors and convergence rates.

(c) Further modify the code to allow $\Delta x \neq \Delta y$ and test the modified script on the domain $(0, 1)^2$ with $\Delta x = 3\Delta y$. Let $h = \max(\Delta x, \Delta y)$. Give the errors and convergence rates for $h = 1/10, 1/20, 1/40, 1/80$.

(2) (30 points)

(a) Consider the finite difference method:

$$\frac{U_{i+2} - U_{i+1} - U_{i-1} + U_{i-2}}{3h^2} = f(x_i)$$

to solve the problem $u''(x) = f(x)$ on a uniform grid $x_i = ih$ of the unit interval (i.e. $(0, 1)$). What is the order of the local truncation error? Is this finite difference method consistent?

(b) Consider the finite difference method:

$$\frac{U_{i+1,j} - U_{i,j+1} - U_{i,j-1} + U_{i-1,j}}{h^2} = f(x_i, y_j)$$

to solve the PDE

$$u_{xx} - u_{yy} = f(x, y)$$

on a uniform grid $x_i = ih, y_j = jh$ of the unit square (i.e. $(0, 1)^2$). What is the order of the local truncation error? Is this finite difference method consistent?

(3) (10 pts) How would you perform the following calculations to avoid cancellation? Justify your answers.

i. Evaluate $\sqrt{x + 1} - 1$ for $x \approx 0$.

ii. Evaluate $\sin(x) - \sin(y)$ for $x \approx y$.

iii. Evaluate $\frac{1 - \cos(x)}{\sin(x)}$ for $x \approx 0$.

(4) (10 pts) Consider the polynomial $p(x) = (x - 2)^9 = x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512$.

i. Plot $p(x)$ for $x = 1.920, 1.921, 1.922, \ldots, 2.080$ (i.e. $x = [1.920 : 0.001 : 2.080]$) evaluating $p$ via its coefficients.

ii. Produce the same plot again, now evaluating $p$ via the expression $(x - 2)^9$.

iii. What is the difference? What is causing the discrepancy? Which plot is correct?

(5) (20 points) This problem is pledged! You may not discuss this with anyone but your instructor. You may not consult any source other than approved textbooks, CAAM 452 lecture notes or your in-class notes to help you with the problem.
Take this opportunity to explore Chebyshev polynomials
i. Find the roots of the Chebyshev polynomial $T_n(x)$.
ii. Show that the closest root to the end points ($x = \pm 1$) is approximately $\frac{1}{n^2}$ away.
iii. Derive the 3 term recursion which expresses $T_n(x)$ as polynomials.

**Challenge problem**

Find the map that takes you from interpolating using Chebyshev polynomials to interpolating with the Cardinal/Lagrange basis. *Hint:* See Boyd.