CAAM 536
Homework 3
Due Feb 25

(1) (20 points) If \( x_0, x_1, \ldots, x_N \in \mathbb{R} \) are distinct, then the cardinal function \( C_j(x) \) denoted by

\[
C_j(x) = \frac{1}{a_j} \prod_{k=0}^{N} (x - x_k), \quad a_j = \prod_{k=0}^{N} (x_j - x_k)
\]

is the unique polynomial interpolant of degree \( N \) to the values 1 at \( x_j \) and 0 at \( x_k, k \neq j \). Take the logarithm and differentiate to obtain

\[
C'_j(x) = p_j(x) \sum_{k=0}^{N} (x - x_k)^{-1}
\]

and from this derive the formulas

\[
D_{ij} = \frac{1}{a_j} \prod_{k=0}^{N} (x_i - x_k) = \frac{a_i}{a_j(x_i - x_j)} \quad (i \neq j)
\]

and

\[
D_{jj} = \sum_{k=0}^{N} (x_j - x_k)^{-1}
\]

for entries of the \( N \times N \) differentiation matrix associated with the points \( \{x_j\} \).

(2) (10 points) Modify cheb so that it computes the diagonal entries of \( D_N \) by the explicit formulas

\[
(D_N)_{00} = \frac{2N^2 + 1}{6}, \quad (D_N)_{NN} = -\frac{2N^2 + 1}{6}
\]

\[
(D_N)_{jj} = -\frac{x_j}{2(1 - x_j^2)} \quad j = 1, \ldots, N - 1
\]

rather than by

\[
(D_N)_{ii} = -\sum_{j=0}^{N} \sum_{j \neq i} (D_N)_{ij}.
\]

Confirm that your code produces the same results except for rounding errors. Then see if you can find numerical evidence that it is less stable numerically than cheb.

(3) (40 points) This problem is pledged! You may not discuss this with anyone but your instructor. You may not consult any source other than approved textbooks, CAAM 452 lecture notes or your in-class notes to help you with the problem.

(a) Implement the spectral method of second order for solving

\[
u''(x) + cu(x) = f(x), \quad 0 < x < 1
\]

\[
u(0) = \alpha, \quad u(1) = \beta
\]

where \( c = 1 \) and the exact solution is \( u(x) = xe^{-x} \). Make a table of the max-norm error vs. \( N \) the order of the discretization for \( N = 10, 20, 40, 80 \). Discuss the
performance of the method compared to the finite difference method (HW1) with the same number of discretization points.

(b) Solve the boundary value problem

\[ u_{xx} + 4u_x + e^x u = \sin(8x) \]

numerically on \([-1, 1]\) with boundary conditions \(u(\pm 1) = 0\). To ten digits of accuracy, what is \(u(0)\)?

(4) (30 points) The matlab script poisson_spec.m solves the Poisson problem on the unit square \((0,1)^2\) using the spectral method. It is set up to solve a test problem for which the exact solution is \(u(x,y) = \exp(x + y/2)\), using Dirichlet boundary conditions and the right hand side \(f(x,y) = 1.25\exp(x + y/2)\).

(a) Test this script increasing the order of the discretization: \(N = 10, 20, 40, 80\). Does this perform as expected? How does it compare with the finite difference method?

(b) Modify the script so that it works on the domain \([a_x, b_x] \times [a_y, b_y]\). Test your modified script on the domain \((0,1) \times (1,2)\) and use the values \(N = 10, 20, 40, 80\). Give the errors.