(1) (50 points)
(a) Implement the finite element method using continuous piecewise linears for solving
\[-u''(x) = f(x), \quad 0 < x < 1\]
\[u(0) = u(1) = 0\]
(b) Verify your code on the following two examples
1. The exact solution is \[u(x) = x(1 - x)\].
2. The exact solution is \[u(x) = x(1 - x)e^{-x}\].
Consider a sequence of uniform meshes with \(h = 1/4, 1/8, 1/16, 1/32\). Denote by \(uh\) the finite element solution. For each mesh, plot the numerical solution and compute the following errors:
\[err_0 = (\int_0^1 (u - uh)^2 dx)^{1/2}, \quad err_1 = (\int_0^1 (u' - uh')^2 dx)^{1/2}.\]
Obtain the numerical convergence rates for \(err_0\) and \(err_1\). Hint: to compute the errors, write the integral as a sum of integrals over each subinterval and use a quadrature rule (trapezoid rule for \(err_0\) and midpoint rule for \(err_1\)). For instance if \(h = 1/(N + 1)\) and the grid nodes are \(x_i = ih\), we can write
\[\int_0^1 (u(x) - uh(x))^2 dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} (u(x) - uh(x))^2 dx \approx \sum_{i=1}^{N} \frac{h}{2} \left((u(x_{i-1}) - uh(x_{i-1}))^2 + (u(x_i) - uh(x_i))^2\right).\]