What is a partial differential equation (PDE)?

PDEs are the equations that are used to model physical phenomena. Examples include climate and weather modeling, blood flow, medical imaging, and seismic imaging. Some PDEs can be solved analytically but the majority of them can not. Thus numerical methods are required to approximate the solutions.

1. General definitions

Definition 1.1. A partial differential equation (PDE) is an equation relating an unknown function $u(x_1, \ldots, x_n)$ with independent variables $x_1, \ldots, x_n$ and a finite number of partial derivatives of $u$; i.e.

$$ F \left( x_1, \ldots, x_n, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \ldots, \frac{\partial^k u}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} \right) = 0 $$

where $\sum_{j=1}^{n} k_j = k$.

Definition 1.2. The order of the partial differential equation is that of the highest derivative.

Definition 1.3. A partial differential equation is called quasilinear if it is linear in all the highest order derivatives of the unknown function $u$.

Example 1.1.

$$ A(x, y, u, u_x, u_y) \frac{\partial^2 u}{\partial x^2} + B(x, y, u, u_x, u_y) \frac{\partial^2 u}{\partial x \partial y} + C(x, y, u, u_x, u_y) \frac{\partial^2 u}{\partial y^2} + F(x, y, u, u_x, u_y) = 0 $$

is a second order quasilinear PDE.

Definition 1.4. A PDE is called linear if it is linear in the unknown function and its partial derivatives.

Example 1.2.

$$ A(x, y) \frac{\partial^2 u}{\partial x^2}(x, y) + B(x, y) \frac{\partial^2 u}{\partial x \partial y}(x, y) + C(x, y) \frac{\partial^2 u}{\partial y^2}(x, y) + D(x, y) \frac{\partial u}{\partial x}(x, y) + E(x, y) \frac{\partial u}{\partial y} + G(x, y) u(x, y) = F(x, y) $$

is a second order linear PDE.

2. Classification of second order PDEs

(This is from Colton’s PDE book) Consider the second order PDE

$$ \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + f \left( x_1, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right) = 0 $$

where $a_{ij} = a_{ij}(x_1, \ldots, x_n)$ are given real valued functions defined on a domain $D \subset \mathbb{R}^n$ and without loss of generality assume that $a_{ij} = a_{ji}$. Let $(x_1^0, \ldots, x_n^0)$ be a fixed point in $\mathbb{R}^n$. Consider the quadratic form

$$ \sum_{i,j=1}^{n} a_{ij}(x_1^0, \ldots, x_n^0) t_i t_j. $$

- Equation (1) is called elliptic at $(x_1^0, \ldots, x_n^0)$ if at this point (2) can be reduced via a real linear transformation to the sum of $n$ squares all of the same sign.
- Equation (1) is called hyperbolic at $(x_1^0, \ldots, x_n^0)$ if at this point (2) can be reduced via a real linear transformation to the sum of $n$ squares of the squares of which $n-1$ are of the same sign.
• Equation (1) is called parabolic at \((x_0^1, \ldots, x_0^n)\) if at this point (2) can be reduced via a real linear transformation to the sum of fewer than \(n\) squares not necessarily of the same sign.

**Example 2.1.** The wave equation

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial t^2}
\]

is hyperbolic in any domain.

**Example 2.2.** The Laplace equation

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0
\]

is elliptic in any domain.

**Example 2.3.** The heat equation

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t}
\]

is parabolic in any domain.

3. **Boundary value problems**

In this class, we will focus on numerical techniques for approximating solutions to elliptic boundary value problems.

**Definition 3.1.** A PDE coupled with a boundary condition is called a boundary value problem.

**Example 3.1.** Let \(D \subset \mathbb{R}^3\). Then

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f(x, y, z) \quad (x, y, z) \in D
\]

\[
u(x, y, z) = g(x, y, z) \quad (x, y, z) \in \partial D
\]

where \(\partial D\) denotes the boundary of \(D\) is a Dirichlet boundary value problem.

If \(f(x, y, z) \neq 0\), the PDE is called the Poisson equation.

**Example 3.2.** Let \(D \subset \mathbb{R}^3\). Then

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f(x, y, z) \quad (x, y, z) \in D
\]

\[
\frac{\partial u}{\partial \nu}(x, y, z) = g(x, y, z) \quad (x, y, z) \in \partial D
\]

where \(\partial D\) denotes the boundary of \(D\) and \(\nu\) is the outward facing normal vector is a Neumann boundary value problem.

**Example 3.3.** Let \(D \subset \mathbb{R}^3\). Then

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f(x, y, z) \quad (x, y, z) \in D
\]

\[
\frac{\partial u}{\partial \nu}(x, y, z) + \eta u(x, y, z) = g(x, y, z) \quad (x, y, z) \in \partial D
\]

where \(\partial D\) denotes the boundary of \(D\), \(\eta\) is some constant and \(\nu\) is the outward facing normal vector is a Robin boundary value problem.