Functional Spaces

Before we can start talking about finite element methods we need to introduce some functional analysis.

Why? Unlike the methods we have talked about thus far, finite element method do not discretize the strong form of the PDE. Instead they recast the problem in a variational form by looking for a solution in Sobolev spaces (special Hilbert spaces).

We begin by learning about the Sobolev spaces from the top-down. (Most general space they are a subset of them.)

Inner-product Spaces

Def: An inner product space is a real vector space $X$ in which an inner product has been defined. The inner product must satisfy the following properties:

1. $\forall u, v \in X \ (u, v)$ is a real #
2. $\forall u, v \in X \ (u, v) = (v, u)$
3. $\forall u, v \in X \ \forall \alpha \in \mathbb{R} \ (\alpha u, v) = \alpha (u, v)$
4. $\forall u \in X \ (u, u) > 0$ if $u \neq 0$
5. $\forall u, v, w \in X \ ((u + v, w) = (u, w) + (v, w)$
Ex 1) \( X = \mathbb{R}^n \) Let \( u, v \in X \).
\[
(u, v) = u^T v = u \cdot v
\]

2) \( X = L^2(\mathbb{R}) \) for any domain \( \Omega \subset \mathbb{R}^2 \)

\[
L^2(\mathbb{R}) = \{ \text{Lebesgue measurable function } u: \Omega \to \mathbb{R} \} \text{ s.t. } \int \Omega |u|^2 \, dA < \infty
\]

Let \( u, v \in X \) then
\[
(u, v) = \int_{\Omega} u(x)v(x) \, dx
\]

**Def.** If \((u, v) = 0\) then we say that \( u \) and \( v \) are orthogonal in \( X \).

Let \( v \in X \). If \( u \perp v \) then \( u \perp v \).

**Lemma:** An inner-product space is normed space with the norm:
\[
\|u\| = \sqrt{(u, u)}
\]

This norm satisfies the following properties:

1. \( \|u\| = 0 \) if \( u = 0 \)
2. \( \|u\| > 0 \) if \( u \neq 0 \)
3. \( \|au\| = |a| \|u\| \) for \( a \in \mathbb{R} \)
4. \( \|u + v\| \leq \|u\| + \|v\| \) triangle inequality
5. Pythagorean Thm. If \( (u, v) = 0 \) then \( \|u + v\|^2 = \|u\|^2 + \|v\|^2 \)
6. Cauchy-Schwarz inequality
\[
|u, v| \leq \|u\| \|v\|
\]
Proof of Cauchy-Schwarz inequality

Let \( \lambda \in \mathbb{R} \). Then

\[
0 \leq \| u - \lambda v \|^2 = (u - \lambda v, u - \lambda v) = \| u \|^2 - 2\lambda(v, u) + \lambda^2\| v \|^2 = f(\lambda)
\]

This is a quadratic in terms of the variable \( \lambda \).

Note this parabola is facing up. Thus we can look at its minimum which should be greater than or equal to 0.

Recall \( f(\lambda) = a\lambda^2 + b\lambda + c \)

\[
= a(\lambda^2 + \frac{b}{a}\lambda) + c
= a(\lambda^2 + \frac{b}{a}\lambda + \left(\frac{b}{2a}\right)^2) + c - \frac{b^2}{4a}
= a(\lambda + \frac{b}{2a})^2 + c - \frac{b^2}{4a}
\]

Thus \( c - \frac{b^2}{4a} \geq 0 \).

For our quadratic polynomial \( a = \| v \|^2 \), \( b = -2(v, u) \), \( c = \| u \|^2 \)

Thus

\[
\| u \|^2 - \left( \frac{4 (v, u)^2}{\| u \|^2} \right) \geq 0 \quad \text{i.e.} \quad |(v, u)| \leq \| u \| \| v \|
\]

**NOTE:** An alternate proof is given in CAAM 458 Notes.
Hilbert Spaces

Def: A Hilbert space is a complete inner product space.

Def: If every Cauchy sequence in a space $X$ converges to a pt in $X$ then the space is complete.

Def: A sequence $\{u_n\}_{n=0}^{\infty}$ in a normed space is said to be Cauchy if $\forall \varepsilon > 0$, $\exists N_0 \in \mathbb{N}$ such that $\forall m, n \geq N_0$ $\|u_n - u_m\| \leq \varepsilon$.

i.e. $\limsup_{n \to \infty} \|u_i - u_j\| = 0$.

Ex: Consider the sequence $\{x_n\}_{n=1}^{\infty}$: $x_n = \frac{1}{n}$.

Prove that $\{x_n\}$ is a Cauchy sequence.

Sohh: Pick $\varepsilon > 0$. Then our goal is to find (if possible) an $N \in \mathbb{N}$ such that $\forall m, n \geq N$ $|x_n - x_m| < \varepsilon$.

Let's take a closer look.

$|x_n - x_m| = \left| \frac{1}{n} - \frac{1}{m} \right| < \frac{1}{n} + \frac{1}{m} < \frac{2}{N}$

So we need $N > \frac{2}{\varepsilon}$.

$\therefore$ for any $\varepsilon$, pick $N > \frac{2}{\varepsilon}$ then $\forall m, n \geq N$ $|x_n - x_m| < \varepsilon$.

Therefore $\{x_n\}$ is a Cauchy sequence. Note $\{x_n\}$ is not complete.

Lemma: $l^2(\mathbb{N})$ is a Hilbert space.
Dual spaces

**Def.** Let $X$ be an inner product space. Let $\Phi : X \to \mathbb{R}$ be a linear mapping. Then we say $\Phi$ is a **linear functional**.

- $\Phi$ is called a **bounded linear functional** if
  $$\sup_{u \in X, \|u\| = 1} |\Phi(u)| < \infty$$
- The norm of $\Phi$ is defined as
  $$\|\Phi\| = \sup_{u \in X, \|u\| = 1} |\Phi(u)|$$
- The space of all bounded linear functionals is called the **dual space** of $X$. We denote it by $X'$.

**Lemma:** A linear functional is continuous iff it is bounded.

**Ex:** Let $X$ be an inner product space. Fix $u \neq 0$ in $X$.
Define the mapping $\Phi : X \to \mathbb{R}$ by $\Phi(v) = (u, v)$.
Then $\Phi \in X'$ and one can show $\|\Phi\| = \|u\|$
Thus if $X$ is bounded then so is $\Phi$.

**Reisz Representation Thm**

Let $X$ be a Hilbert space and let $\Phi \in X'$. Then there is a unique $u \in X$ s.t. $\forall v \in X \quad \Phi(v) = (u, v)$

i.e. In a Hilbert space, you can write any bounded linear functional as an inner-product functional
Weak derivatives

**Def:** A subset $K \subseteq \mathbb{R}^n$ is compact iff $K$ is closed and bounded.

- Let $\Omega \subseteq \mathbb{R}^n$ be an open domain. Let $U : \Omega \to \mathbb{R}$.
  
  The support of $u$ is $\text{supp}(u) = \{ x \in \Omega : u(x) \neq 0 \}$
  i.e. closure of the set of $x \in \Omega$ st $u(x) \neq 0$.

- Let $C_0^\infty$ be the set of all function that are infinitely differentiable in $\Omega$ and that have compact support.

**Note:** All function in $C_0^\infty(\Omega)$ & their partial derivatives of any order are 0 on the boundary of $\Omega$.

**Ex:** $\phi_{x_0}(x) = \begin{cases} e^{-\left(\frac{x^2}{2} + 1 - x \cdot x_0\right)} & \text{for } \exists x : |x - x_0|^2 < r^2 \\ 0 & \text{elsewhere} \end{cases}$

for any pt $x_0 \in \Omega$ & $r > 0$ st $\exists x : |x - x_0|^2 < r^2$ for $x_0 \in C_0^\infty(\Omega)$.

**Def:** We say $u$ is locally integrable on $\Omega$ if for any $K \subseteq \Omega$ with compact $\text{supp} u < \infty$.

**Notation** $u \in L^1_{loc}(\Omega)$

Let's explore a little. Suppose $u \in C^1(\Omega)$. Pick $\phi \in C_0^\infty(\Omega)$

Then $\int_\Omega \frac{\partial u}{\partial x} \phi \, dA = \left. \phi u \right|_{\partial \Omega} - \int_\Omega u \frac{\partial \phi}{\partial x} \, dA$

via integration by parts. Since $\phi \in C_0^\infty(\Omega)$, $\left. \phi u \right|_{\partial \Omega} = 0$.

functional
Thus \( \int_{\Omega} \frac{du}{dx} \phi \, dA = - \int_{\Omega} u \frac{d\phi}{dx} \, dA \).

This leads us to the formal definition of weak derivative.

**Def:** Let \( u \) be locally integrable on \( \Omega \). The **weak partial derivative** of \( u \) **wrt** \( x \) is the function \( g \in L^1_{\text{loc}}(\Omega) \) that satisfies \( \forall \phi \in C_0^\infty(\Omega) \)

\[
\int_{\Omega} g \phi \, dA = - \int_{\Omega} u \frac{d\phi}{dx} \, dA.
\]

**Note:** If, in addition, \( u \in C^1(\Omega) \) then the weak derivative \( g \) is equal to the strong derivative \( \frac{du}{dx} \), i.e. \( g = \frac{du}{dx} \) in the classic sense.

**Extension to higher order derivatives**

If \( m, n \) are integers, the weak partial derivatives of order \( m \times n \) of \( u \) is the function \( g \in L^1_{\text{loc}}(\Omega) \) that satisfies \( \forall \phi \in C_0^\infty(\Omega) \)

\[
\int_{\Omega} g \phi \, dA = (-1)^{m+n} \int_{\Omega} u \frac{d^{m+n} \phi}{dx^m dy^n} \, dA
\]

**Notation abuse** \( g := \frac{d^{m+n} u}{dx^m dy^n} \) (This is only true if \( u \in C^{m+n}(\mathbb{R}) \))

**Ex:** Define \( u(x, y) = \begin{cases} x & 0 < x \leq 0.5 \\ 1-x & 0.5 < x < 1 \end{cases} \) on \( \Omega = (0, 1)^2 \). Derive the weak derivative of \( u \) wrt \( x \).
Sln: Note $u$ is clearly not differentiable w.r.t $x$. But it is weakly differentiable.

Let $\phi \in C_c^\infty (\Omega)$. Then look at

$$\int_{\Omega} u \frac{\partial \phi}{\partial x} \, dx \, dy = \int_{\Omega} \phi \frac{\partial u}{\partial x} \, dx \, dy$$

$$= \int_{\Omega} \phi \, dx \, dy$$

$$= \int_0^1 \int_0^{y_2} (1-x) \phi \, dx \, dy$$

$$= \int_0^1 \left( \int_0^{y_2} \phi \, dx \right) \, dy$$

$$= \int_0^1 \left( \int_0^{y_2} \left( \phi(0, y) - \phi(y_2, y) \right) \, dx \right) \, dy$$

$$= \int_0^1 \left( \int_0^{y_2} \phi \, dx \right) \, dy$$

$$= \int_0^1 \left( \int_0^{y_2} \phi \, dx \right) \, dy$$

$$\Rightarrow g(x, y) = \begin{cases} 1 & 0 < x \leq \frac{y_2}{2} \\ -1 & \frac{y_2}{2} < x \leq 1 \end{cases} \quad \forall (x, y) \in (0, 1)^2$$

Note we must verify that $g \in L^1_{loc} (\Omega)$ before we can conclude it is the weak derivative of $u$. (It is clear that this is true.)
**Sobolev Spaces**

Recall:
- \( L^2(\Omega) = \{ u \text{ measurable } : \int_\Omega u^2 \, dA < \infty \} \)
- Gradient of \( u \) is a vector defined by \( \nabla u = (\frac{du}{dx}, \frac{du}{dy}) \) on 2-dimensional domains

For the purposes of this discussion, these are **weak derivatives**.

**Def.** The Sobolev space \( H^1(\Omega) \) consists of the set of functions \( u \in L^2(\Omega) \) such that \( \nabla u \in L^2(\Omega) \).

i.e.

\[
\int_\Omega (\frac{du}{dx})^2 \, dA < \infty \quad \& \quad \int_\Omega (\frac{du}{dy})^2 \, dA < \infty.
\]

In set notation,

\( H^1(\Omega) = \{ u \in L^2(\Omega) : \nabla u \in L^2(\Omega) \} \)

By increasing the order of the weak partial derivatives we want in \( L^2(\Omega) \), we get other Sobolev spaces.

**Ex:** \( H^2(\Omega) = \{ u \in H^1(\Omega) : \frac{\partial^2 u}{\partial x^m \partial y^n} \in L^2(\Omega) \text{ for } 0 \leq m+n \leq 2 \} \)

What does say about the relationship of all these Sobolev spaces? They are nested.

\( H^k(\Omega) \subset H^{k-1}(\Omega) \subset \ldots \subset H^2(\Omega) \subset H^1(\Omega) \subset L^2(\Omega) \).

**Note:** Sobolev spaces are Hilbert spaces

**Ex:** \( H^1(\Omega) \) has the following inner product for \( u, v \in H^1(\Omega) \)

\[
(u,v) = \int_\Omega (u \cdot v + \nabla u \cdot \nabla v) \, dA
\]

and the norm

\[
\| u \|_{H^1(\Omega)} = \left( \int_\Omega ((u \cdot \nabla u + u^2) \, dA) \right)^{1/2}
\]

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Functional \( @ \)
For PDEs we need an additional Sobolev space
\[ H_0^1(\Omega) = \{ v \in H^1(\Omega) : \nabla v = 0 \text{ on } \partial \Omega \} \]
One can show that with this restriction to the boundary the inner product on the space is
\[ (u, v) = \int_\Omega \nabla u \cdot \nabla v \, dA \quad \text{with norm } \|u\|_{H_0^1(\Omega)} = \left( \int_\Omega (\nabla u)^2 \, dA \right)^{1/2} \]

Green's Theorem

- Let \( u \in H^2(\Omega) \) and \( v \in H^1(\Omega) \)
  \[
  - \int_\Omega (\nabla u) v \, dA = \int_\Omega \nabla u \cdot \nabla v \, dA + \int_{\partial \Omega} (\nabla u \cdot \mathbf{n}) v \, dl
  \]
  where \( \mathbf{n} \) is the outward facing normal vector on \( \partial \Omega \).

- Let \( k \) be a symmetric positive definite matrix. Then
  \[
  - \int_\Omega (\nabla (k \nabla u)) v \, dA = \int_\Omega k \nabla u \cdot \nabla v \, dA - \int_{\partial \Omega} (k \nabla u \cdot \mathbf{n}) v \, dl
  \]

Poincaré Inequality

There exists a constant \( C_p > 0 \) such that for all \( v \in H_0^1(\Omega) \),
\[ \|v\|_{L^2(\Omega)} \leq C_p \|\nabla v\|_{L^2(\Omega)} = \|v\|_{H_0^1(\Omega)} \]