Consider \( \Delta u = u_{xx} + u_{yy} = f(x,y) \) \( x \in (-1,1)^2 \) \( y \in \mathbb{R} \).

\[ u(x,y) = g(x,y) \quad x \in \mathbb{R} \]

How are we going to choose our basis? We will assume everything is separable. (This is!)

3 use a product basis.

\[ \phi_{m,n}(x,y) = \phi_m(x) \phi_n(y) \quad m = 0, \ldots, N_x \]
\[ n = 0, \ldots, N_y \]

where \( \phi_m, \phi_n \) are the cardinal basis w/ pts \( x_i \) \( i = 0, N_x \) \( y_j \) \( j = 0, \ldots, N_y \)

\[ \mathcal{D} (x_i, y_j) : i = 0, \ldots, N_x \]
\[ j = 0, \ldots, N_y \]

This is called the tensor product grid!

Plot on computer.

By doing this we can write everything into linear 1-D operators.

To deal with the tensor product grid via linear algebra we will use Kronecker products.

\[ A \otimes B = \text{Kron}(A,B) \text{ in } \text{matlab.} \]

\[ \text{Kronecker product of } A \otimes B \]

\[ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \ 2a & b \ 2b \\ c \ 2c & d \ 2d \end{pmatrix} \]

\[ \begin{pmatrix} 3a \ 3b & 4a \ 4b \\ 3c \ 3d & 4c \ 4d \end{pmatrix} \]
How are we going to order pts:

since we know the solution on the boundary.

We want to take derivatives w.r.t. $x$ by directions independently.
So let's do an example where we build the 2nd derivative spectral operators for our problem.

Ex: let $N = 4$.

\[
\begin{align*}
D &= \text{Cheb}(4); \\
D_2 &= D^2; \\
D_2^2 &= D_2(2:4, 2:4). \\
\hat{D}_N^2 &= \text{3x3 differentiation matrix.}
\end{align*}
\]

1st $x$-derivatives $D_2 x = I \otimes \hat{D}_N^2$ \\
2nd $y$-derivatives $D_2 y = \hat{D}_N^2 \otimes I$

\[
\Rightarrow \text{The discrete Laplacian is } \\
L_N = I \otimes \hat{D}_N^2 + \hat{D}_N^2 \otimes I.
\]

Recall eigenvalues of $\hat{D}_N^2$ are \[ \lambda = -\pi^2/4 \]

\[ V = \sin(n \pi (x_1 + x_2)/2) \]
Dirichlet BVP.

\[ \Delta u = f(x), \quad x \in (-1,1)^2 = \Omega \]
\[ u(x) = g(x), \quad x \in \Gamma = \partial \Omega \]

Discretize via spectral method.

\[ \Delta u \approx \sum_{n=2}^{N} + D_n^2 \cdot I = \lambda u \]

We know \( u(x) \) for \( x \in \Gamma \).
We can reorder pts.

let \( I_i = \sum x_j, \quad j=1,N^2: x_j \in \Omega \)
let \( I_e = \sum x_j, \quad j=1,N^2: x_j \in \Gamma \).

Then we can solve the system

\[
\begin{bmatrix}
I_n \quad (I_i, \cdot) \\
L_n(I_i, \cdot)
\end{bmatrix}
\begin{bmatrix}
\dot{u}_i \\
\ddot{u}_i
\end{bmatrix}
= \begin{bmatrix}
g(x_e) \\
f(x_e)
\end{bmatrix}
\]

A.