CAAM 540 · APPLIED FUNCTIONAL ANALYSIS

Problem Set 1

Posted Tuesday 26 August 2008. Due Tuesday 2 September 2008.

Young, problems 1.1, 1.2, 1.3, 1.11, 2.4 [20 points each]

1.1 For which $s \in \mathbb{C}$ does the sequence $(n^{-s})_{n=1}^{\infty}$ belong to ℓ^2?

1.2 Let $a < b$ in $\mathbb{R}$. Show that the space $W[a,b]$ of continuously differentiable functions on $[a,b]$, with values in $\mathbb{C}$, is an inner product space with respect to pointwise addition and scalar multiplication, and inner product

$$(f,g)_W = \int_a^b f(t)\overline{g(t)} + f'(t)\overline{g'(t)}\,\mathrm{d}t.$$

1.3 A *trigonometric polynomial* is a function of the form

$$f(x) = \sum_{n=1}^{k} a_n \mathrm{e}^{\mathrm{i}\lambda_n x}$$

where $k \in \mathbb{N}$, $a_1, \ldots, a_k \in \mathbb{C}$, and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$. The space TP of trigonometric polynomials is a vector space over $\mathbb{C}$ with respect to pointwise addition and scalar multiplication. Prove that the formula

$$(f,g) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\overline{g(x)}\,\mathrm{d}x$$

defines an inner product on TP.

1.11 Let $k_\alpha \in RL^2$ be defined by

$$k_\alpha(z) = (1 - \overline{\alpha} z)^{-1},$$

where $|\alpha| \neq 1$. Show that, for $f \in RH^2$,

$$(f, k_\alpha) = \begin{cases} f(\alpha) & \text{if } |\alpha| < 1 \\ 0 & \text{if } |\alpha| > 1. \end{cases}$$

From Young, page 9:
RL^2 denotes the space of rational functions which are analytic on the unit circle

$$\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\},$$

with the usual addition and scalar multiplication and with the inner product

$$(f,g) = \frac{1}{2\pi\mathrm{i}} \int_{\partial\mathbb{D}} f(z)\overline{g(z)}\,\frac{\mathrm{d}z}{z}, \qquad (*)$$

the integral being taken counter-clockwise around ∂D.
RH^2 is the subspace of RL^2 consisting of those rational functions which are analytic on the closed unit disc $\operatorname{clos}\mathbb{D}$, where

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

with inner product given by $(*)$.

2.4 Prove that there is no inner product on $C[0,1]$ such that $(f,f)^{1/2} = \|f\|_\infty$ for all f.
(Show that the parallelogram law does not hold for $\|\cdot\|_\infty$.)