CAAM 540 · APPLIED FUNCTIONAL ANALYSIS

Problem Set 3

Posted Tuesday 9 September 2008. Due Tuesday 16 September 2008.

Pick any four of Problems 3.10, 3.x, 4.2, 4.7, 4.9 from Young [25 points each]

3.10 Let A be a non-empty closed convex set in a Hilbert space. Show that A contains a unique vector a of smallest norm and that $\mathrm{Re}(a, a - x) \le 0$ for all $x \in A$.

3.x A Banach space B is *uniformly convex* provided that for all $\varepsilon > 0$, there exists some $\delta > 0$ such that $\|\frac{1}{2}(f+g)\| > 1 - \delta$ implies $\|f - g\| < \varepsilon$ for all $f, g \in B$ with $\|f\| = \|g\| = 1$.

(a) Prove that all Hilbert spaces are uniformly convex.

(b) Give an example of a Banach space that is not uniformly convex.

(c) Let S be a convex subset of a uniformly convex Banach space B, and suppose there exists at least one point $s \in S$ closest to some $x \in B$: $\|x - s\| = \inf_{r \in S} \|x - r\|$. Prove that s must be unique.

4.2 Let $e_j(z) = z^j$ for $z \in \mathbb{C}$, $j \in \mathbb{Z}$. Show that $(e_j)_{j=-\infty}^{\infty}$ is an orthonormal sequence in RL^2.

4.7 The first three Legendre polynomials are

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \tfrac{1}{2}(3x^2 - 1).$$

Show that the orthonormal vectors in $L^2(-1, 1)$ obtained by applying the Gram–Schmidt process (see, e.g., Problem 4.6) to $1, x, x^2$ are scalar multiples of these.

4.9 Find

$$\min_{a,b,c \in \mathbb{C}} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2 \, \mathrm{d}x.$$

(Use the solution of Problem 4.7).