CAAM 540 · APPLIED FUNCTIONAL ANALYSIS

Problem Set 4

Posted Saturday 27 October 2008. Due Friday 3 October 2008.

Young, problems 4.x, 5.1(a,c), 5.4, 6.8 [25 points each]

4.x Suppose that $\{e_n\}$ and $\{f_n\}$ are orthonormal sequences in a Hilbert space, and assume that $\{e_n\}$ is complete. Show that if

$$\sum \|e_n - f_n\|^2 < 1,$$

then $\{f_n\}$ is also complete. [Hutson & Pym]

5.1 Find the Fourier series of the following functions on $[-\pi, \pi]$ and use Corollary 5.7 to deduce the stated formulae.

(a) $f(x) = x$; $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(c) $f(x) = \mathrm{e}^{sx}$; $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + s^2} = \frac{\pi}{s} \coth\ \pi s$.

[To appreciate the utility of this approach, you might try computing $\sum_{n=1}^{\infty} n^{-2}$ by more direct means.]

5.4 Let

$$g_n(x) = \frac{\mathrm{e}^{inx}}{\sqrt{2\pi(1+n^2)}}, \quad n \in \mathbb{Z}, -\pi \le x \le \pi.$$

Show that $(g_n)_{-\infty}^{\infty}$ is an orthonormal sequence in $W[-\pi, \pi]$.

Let

$$W_0 = \{f \in W[-\pi, \pi] : f(\pi) = f(-\pi)\}.$$

Show that sinh is orthogonal to every function in W_0. Deduce that $(g_n)_{-\infty}^{\infty}$ is *not* a complete orthonormal sequence in $W[-\pi, \pi]$.

6.8 Let F be a *nonzero* linear functional on a normed space E and let

$$\mathrm{Ker}F = \{x \in E : F(x) = 0\}.$$

Show that Ker$F$ has co-dimension 1 in E (i.e., there exists some fixed $y \in E$ such that any element $x \in E$ can be written in the form $x = z + \gamma y$ for some $z \in \mathrm{Ker}F$ and $\gamma \in \mathbb{C}$).

Deduce that if V is a subspace of E containing Ker$F$ then either $V = E$ or $V = \mathrm{Ker}F$. Hence show that if F is discontinuous then Ker$F$ is dense in E. Conclude that F is continuous if and only if Ker$F$ is closed.