CAAM 540 · APPLIED FUNCTIONAL ANALYSIS

Problem Set 5

Posted Sunday 5 October 2008. Due Friday, 10 October 2008.

Young, 7.9, 7.10, 7.14, 7.36 [25 points each]

7.9 An operator is said to have *rank 1* if its range is one-dimensional. Let T be a bounded operator of rank 1 on a Hilbert space H, and let ψ be a non-zero vector in the range of T. Show that there exists $\phi \in H$ such that

$$Tx = (x, \phi)\psi, \qquad \text{all } x \in H$$

and that

$$\|T\| = \|\phi\|\,\|\psi\|.$$

7.10 An operator K on $L^2(0,1)$ is defined by

$$(Kx)(t) = \int_0^1 \mathrm{e}^{t-s} x(s)\,\mathrm{d}s, \qquad 0 < t < 1.$$

Use the result of Problem 7.9 to show that

$$\|K\| = \sinh 1.$$

7.14 Show that the rank 1 operator T of Problem 7.9 satisfies $T^2 = (\psi, \phi)T$. Hence show that, if $(\psi, \phi) \neq 1$, $I - T$ is invertible and find $(I - T)^{-1}$.

7.36 Use Problem 7.14 to show that the spectrum of the rank 1 operator T of Problem 7.9 is

$$\sigma(T) = \{0, (\psi, \phi)\}$$

(assuming dim $H > 1$).