CAAM 540 · APPLIED FUNCTIONAL ANALYSIS

Problem Set 6

Posted Tuesday 27 October 2008. Due Tuesday 4 November 2008.

Young, 7.40, (7.41 or 7.y), 7.z, 8.13 [25 points each]

7.40 By considering the identity

$$(\lambda I - A)(\lambda I + A) = \lambda^2 I - A^2$$

for any $\lambda \in \mathbb{C}$ and $A \in \mathcal{L}(E)$, E a Banach space, show that $\sigma(A^2) = \{\lambda^2 : \lambda \in \sigma(A)\}$. Deduce that, for any $\lambda \in \sigma(A)$,

$$|\lambda| \le \|A^2\|^{1/2}.$$

7.41 Can you replace 2 by 3 in the previous problem? By $n \in \mathbb{N}$? By -1?
Hint: consider principal roots of unity, i.e., $\mathrm{e}^{2k\pi \mathrm{i}/n}$, $k = 0, \ldots, n-1$.

7.y (Spectral Mapping Theorem) Let p be a degree n polynomial, and suppose $A \in \mathcal{L}(E)$ for a Banach space E. If $p(z) = c_0 + c_1 z + \cdots + c_n z^n$, we define $p(A)$ via

$$p(A) = c_0 I + c_1 A + \cdots + c_n A^n.$$

Prove that $\sigma(p(A)) = p(\sigma(A))$, where

$$p(\sigma(A)) = \{p(\lambda) : \lambda \in \sigma(A)\}.$$

Now let r be a rational function of degree (m, n), i.e., $r(z) = p(z)/q(z)$ with p and q polynomials of degree m and n. Suppose that the poles of r are disjoint from $\sigma(A)$ (why?), and define $r(A) = p(A)q(A)^{-1}$. Prove that $\sigma(r(A)) = r(\sigma(A))$.

7.z Banded Laurent operators correspond to doubly-infinite matrices acting on $\ell^p(\mathbb{Z})$ with the constant a_k on the kth diagonal for $-m \le k \le n$, with $0 \le m, n < \infty$ and zero elsewhere. One such example is the doubly-infinite shift operator, which is zero everywhere except for ones on the first superdiagonal,

$$C = \begin{pmatrix} \ddots & \ddots & \ddots & & & \\ \ddots & 0 & 1 & \ddots & & \\ & \ddots & 0 & 1 & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

where we can take $m = 0$ and $n = 1$ with $a_0 = 0$, $a_1 = 1$. If $x = (\ldots, x_{-1}, x_0, x_1, \ldots) \in \ell^p(\mathbb{Z})$, then

$$Cx = C(\ldots, x_{-1}, x_0, x_1, \ldots) = (\ldots, x_0, x_1, x_2, \ldots).$$

(a) Show that for C acting on $\ell^\infty(\mathbb{Z})$, $\sigma(C) = \{z \in \mathbb{C} : |z| = 1\}$, and that all $z \in \sigma(C)$ are eigenvalues. Are these same z eigenvalues if C acts on $\ell^p(\mathbb{Z})$ for $1 \le p < \infty$?

(b) Now consider general banded Laurent operators on $\ell^\infty(\mathbb{Z})$ determined by the coefficients $a_{-m}, \ldots, a_n$:

$$A = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ \ddots & a_{-m} & \ddots & a_{-1} & a_0 & a_1 & \ddots & a_n & \ddots & & & \\ & \ddots & a_{-m} & \ddots & a_{-1} & a_0 & a_1 & \ddots & a_n & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \end{pmatrix}.$$

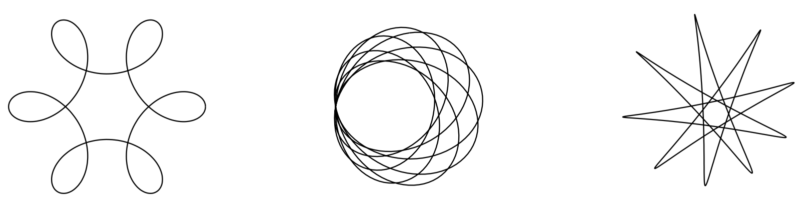
Use the result of Problem 7.y (whether you solved that problem or not) to show that

$$\sigma(A) = \{a(z) : |z| = 1\},$$

where a is the *symbol* of the operator, defined by the Laurent polynomial

$$a(z) = \sum_{k=-m}^{n} a_k z^k.$$

(c) Plot the spectrum $\sigma(A) = \{a(z) : |z| = 1\}$ for a few interesting symbols of your choosing (e.g., using MATLAB). Below are several examples to whet your appetite.



8.13 Let K be a compact Hermitian operator on a Hilbert space H. Show that if K has infinite rank then the range of K is not closed in H.

— Optional Supplemental Problems —

7.27 Let V be the Volterra operator V from Problem 7.8:

$$(Vx)(t) = \int_0^t x(s)\,\mathrm{d}s, \quad 0 < t < 1.$$

Show that $V + V^*$ has rank 1.

7.x Suppose A is a bounded linear operator on the Hilbert space H. Show that

$$\tfrac{1}{2}\|A\| \le \sup_{\|x\|=1} |(Ax, x)|.$$

(You may find it useful to write A as the sum of Hermitian and skew-Hermitian operators.)

Use this result to show that if $(Ax, x) = (Bx, x)$ for all $x \in H$, then $A = B$.

An operator $A \in \mathcal{L}(H)$ is called *normal* provided A commutes with its adjoint, i.e., $A^*A = AA^*$. Show that A is normal if and only if $\|A^*x\| = \|Ax\|$ for all x.