CAAM 540 · APPLIED FUNCTIONAL ANALYSIS

Problem Set 7

Posted Thursday 13 November 2008. Due Wednesday 19 November 2008.
[Late papers are due Friday 21 November]

Young, 8.1 [10 points], 8.14 [15 points], 8.16 [35 points], either (both Ex. 8.6 and 8.x) or 8.y [40 points].

8.1 Let E be a Banach space and let $A, B, C \in \mathcal{L}(E)$. Show that if B is compact, then so is ABC.

8.14 Show that a compact Hermitian operator is the limit with respect to the operator norm ('uniform limit') of a sequence of finite rank operators.

8.16 Let H be a separable Hilbert space and let K be a compact operator on H. Let $(\varphi_j)_1^\infty$ be a complete orthonormal sequence of eigenvectors of K^*K and let $(\lambda_j)_1^\infty$ be the sequence of corresponding eigenvalues. (Explain why this sequence always exists.) Show that the formula

$$U\Big(\sum_{j=1}^{\infty} x_j \varphi_j\Big) = \sum_{j=1}^{\infty} x_j \mu_j K\varphi_j,$$

where

$$\mu_j = \begin{cases} \lambda_j^{-1/2} & \text{if } \lambda_j > 0 \\ 0 & \text{if } \lambda_j = 0, \end{cases}$$

defines U as a bounded linear operator on H. What is $\|U\|$? Show further that $K = U(K^*K)^{1/2}$. Deduce, using Problems 8.1 and 8.14 that K is the limit with respect to the operator norm of a sequence of finite rank operators.

Ex. 8.6 Prove that the Volterra operator V on $L^2(0,1)$ defined by

$$(Vx)(t) = \int_0^t x(s)\,\mathrm{d}s, \quad 0 < s < 1$$

is a Hilbert–Schmidt operator, and thus compact.

8.x The Volterra operator in Exercise 8.6 is non-self-adjoint.

(a) Determine the spectrum of V.

(b) What does your answer to part (a) suggest about the possibility of generalizing the Spectral Theorem to non-self-adjoint compact operators?

[You may use the following result: any *nonzero* $\lambda \in \sigma(K)$ must be an eigenvalue for any compact operator K, and $\sup_{\lambda\in\sigma(A)} |\lambda| = \lim_{n\to\infty} \|A^n\|^{1/n}$ for any bounded linear operator A.]

8.y We are interested in approximating the spectrum of the Fredholm integral operator $K : L^2[a,b] \to L^2[a,b]$, defined pointwise for $t \in [a,b]$ by

$$(Ku)(t) = \int_a^b k(t,s)u(s)\,ds.$$

For purposes of this problem, assume that the kernel $k(t,s)$ is sufficiently well-behaved so as not to complicate the calculations that are to follow. Since K is a compact operator, all nonzero points in its spectrum must be eigenvalues. Hence, we can approximate the spectrum by looking for nonzero solutions to the equation

$$Ku = \lambda u. \tag{1}$$

We can approximate an integral over $[a,b]$ via a *quadrature rule* of the form

$$\int_a^b f(s)\,ds \approx \sum_{\ell=1}^{n} \omega_\ell f(s_\ell),$$

where $\omega_1, \ldots, \omega_n$ are the *quadrature weights* and $s_1, \ldots, s_n \in [a,b]$ are the *quadrature nodes.*

With $t_m = s_m$ for $m = 1, \ldots n$, we can approximate the action of the Fredholm integral operator as

$$(Ku)(t_m) \approx \sum_{\ell=1}^{n} \omega_\ell k(t_m, s_\ell)u(s_\ell).$$

Each value of $m = 1, \ldots, n$ thus provides a row in a finite-dimensional approximation of equation (1):

$$\begin{bmatrix} \omega_1 k(t_1,s_1) & \ldots & \omega_n k(t_1,s_n) \\ \vdots & \ddots & \vdots \\ \omega_1 k(t_n,s_1) & \ldots & \omega_n k(t_n,s_n) \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \lambda \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix},$$

which we write as $\mathbf{Ku} = \lambda\mathbf{u}$. The goal of this problem is to (computationally) study how well the n eigenvalues of the $n \times n$ matrix $\mathbf{K}$ approximate (some subset of) the spectrum of K, as a function of the kernel k and quadrature rule.

For all these examples, we shall use the n-point Gauss–Legendre quadrature rules; a MATLAB code to produce the nodes and weights (adapted from a code in Trefethen's *Spectral Methods in MATLAB*) can be found on the class website.

(a) If $[a,b] = [-\pi, \pi]$ and the kernel $k(t,s)$ has the special form $k(t,s) = \kappa(t-s)$, where κ is a continuous, 2π-periodic function, we saw in class (see Young, Example 7.24) the eigenvectors ϕ_m and associated eigenvalues λ_m of K satisfy, for $m \in \mathbb{Z}$,

$$\phi_m(x) = \frac{\mathrm{e}^{imx}}{\sqrt{2\pi}}, \qquad \lambda_m = \int_{-\pi}^{\pi} \kappa(\tau)\mathrm{e}^{-im\tau}\,\mathrm{d}\tau = (\kappa, \mathrm{e}^{im\tau}).$$

Compute the eigenvalues of the matrix $\mathbf{K}$ for modest values of n for the kernels

$$\kappa(t) = 1, \qquad \kappa(t) = \mathrm{e}^{it}, \qquad \kappa(t) = \sin(t).$$

Explain your results in terms of the exact eigenvalues for this problem.

(b) Now consider the kernel

$$\kappa(t) = |\sin(t)|,$$

which gives exact eigenvalues

$$\lambda_m = \begin{cases} \dfrac{4}{1-m^2} & m \text{ even;} \\ 0 & m \text{ odd.} \end{cases}$$

Produce a plot (e.g., `loglog` in MATLAB) showing how the error in eigenvalues λ_0, λ_2, λ_4, and λ_6 decreases with growing value of n (say, $n = 8, 16, \ldots, 512$).

(c) Repeat the experiment in part (b) with the functions

$$\kappa(t) = |\sin(t)|^3, \qquad \lambda_m = \begin{cases} \dfrac{24}{m^4 - 10n^2 + 9} & m \text{ even;} \\ 0 & m \text{ odd;} \end{cases}$$

and

$$\kappa(t) = \mathrm{e}^{\sin(t)}$$

with

$$\begin{aligned} \lambda_0 &= 7.954926521012845274513219 6\ldots \\ \lambda_2 &= -0.852927764164121486998913 5\ldots \\ \lambda_4 &= 0.017197833556865812429919 4\ldots \\ \lambda_6 &= -0.000141300427371349208486 5\ldots. \end{aligned}$$

Speculate about the reason for the different convergence behavior you observe for the three kernels in parts (b) and (c).

(d) Now consider the non-self-adjoint integral operator

$$(K_F u)(t) = \sqrt{\frac{\mathrm{i}F}{\pi}} \int_{-1}^{1} \mathrm{e}^{-\mathrm{i}F(t-s)^2} u(s)\, ds$$

on $L^2[-1,1]$, which arises in a model for light propagating in a laser cavity (see Trefethen and E., *Spectra and Pseudospectra*, §60, and references therein). The parameter F is called the *Fresnel number*.

Produce plots of the spectrum of K_F in the complex plane for (i) $F = 16\pi$ and (ii) $F = 64\pi$. In each case, select the discretization parameter n sufficiently large that the eigenvalues have converged to plotting accuracy, i.e., they do not appear to move when you increase n.