CAAM 540 · APPLIED FUNCTIONAL ANALYSIS

Problem Set 8

Posted Sunday 23 November 2008; corrected 29 November 2008. Due Monday 1 December 2008.
This problem set is *optional*. If completed, it will replace your lowest normal homework grade.

Complete four of the following problems, worth 25 points each.

1. Let A, B, L, and U be bounded self-adjoint operators in a Hilbert space H.
   (a) Suppose that $L \le A \le U$. Show that $\|A\| \le \max\{\|L\|, \|U\|\}$.
   (b) Suppose that $L \le A \le U$ and $L \le B \le U$. Show that $\|A - B\| \le \|L - U\|$.

2. Let A, B, and C be bounded self-adjoint operators in a Hilbert space H.
   Use the operator square root to show the following.
   (a) If A and B are positive and commute, then AB is positive and self-adjoint.
   (b) Suppose C is a positive operator that commutes with A and B, and that $A \le B$.
   Show that $AC \le BC$.

[Hutson and Pym]

3. Let A denote a bounded linear operator on a Hilbert space H, and recall that a point $\lambda \in \sigma(A)$ is in the *residual spectrum* if $(\lambda - A)^{-1}$ exists but $\mathrm{Dom}\,(\lambda - A)^{-1} = \mathrm{Ran}\,(\lambda - A)$ is not dense in H. In class we proved that if A is self-adjoint, then the residual spectrum is empty. Prove that the same holds if A is *normal*, i.e., if $AA^* = A^*A$.

4. Let $A = \mathrm{diag}(a_1, a_2, \ldots)$ be a diagonal infinite matrix that is a bounded operator on $\ell_2(\mathbb{N})$, i.e., $Ae_n = a_n e_n$, and there exists some finite M such that $|a_n| \le M$ for all $n \in \mathbb{N}$.
   (a) What is $\sigma(A)$? Classify all $\lambda \in \sigma(A)$ as belonging to the point spectrum $\sigma_p(A)$ (i.e., λ is an eigenvalue), the continuous spectrum $\sigma_c(A)$ (the resolvent exists and is densely defined, but is unbounded), or the residual spectrum $\sigma_r(A)$ (the resolvent exists but is not densely defined).
   (b) Let Ω denote any compact subset of the complex plane. Describe a bounded linear operator A for which $\sigma(A) = \Omega$.

[Halmos]

5. Consider the discrete Schrödinger operator A on $\ell_2(\mathbb{Z})$ defined so that for $u \in \ell_2(\mathbb{Z})$, the nth entry of Au is given by
$$(Au)_n = u_{n-1} + V(n)u(n) + u_{n+1},$$
that is, A can be regarded as a doubly-infinite matrix with ones on the super- and sub-diagonal, and values of the potential V on the main diagonal. For this problem, consider the *Fibonacci potential*, defined as
$$V(n) = \begin{cases} \lambda, & \text{if } (n/\phi \bmod 1) \ge 1 - 1/\phi; \\ 0, & \text{otherwise,} \end{cases}$$

where $\phi = \frac{1}{2}(1+\sqrt{5})$ is the golden ratio, and λ is some constant. Sütő (1987) showed that the spectrum of this bounded, self-adjoint operator can be approximated in the following manner. Define the family of polynomials $\{p_n\}$ by the recurrence

$$\begin{aligned} p_{-1}(x) &= 2 \\ p_0(x) &= x \\ p_1(x) &= x - \lambda \\ &\vdots \\ p_{n+1}(x) &= p_n(x)p_{n-1}(x) - p_{n-2}(x). \end{aligned}$$

Then $\sigma(A)$ is approximated by the sets

$$\sigma_n = \{x \in \mathbb{R} : |p_n(x)| \leq 2\},$$

which will consist of a series of real intervals, and the quality of the approximation improves as $n \to \infty$.

(a) Write a code (e.g., in Mathematica or MATLAB) to compute σ_n, and plot these sets for $\lambda = 4.1$ and $n = 0, \ldots, 7$.

(b) What computational challenges to you expect to emerge as n gets large?

(c) Compare your σ_n sets for several values of n (say, $n = 4$ and $n = 7$) with the spectrum you get from taking a 500×500 finite section of the operator A. (To build an $N \times N$ finite section, build an $N \times N$ matrix A_N whose main, super-, and sub-diagonals are drawn from those of A. One mimics the doubly infinite structure of A by making the finite section *periodic*: set the corner entries $A_N(N, 1) = A_N(1, N) = 1$.)