CAAM 540 · APPLIED FUNCTIONAL ANALYSIS

Pledged Problem Set 1

Posted Saturday 18 October 2008. Due 5pm, Friday 24 October 2008.

This problem set is to be pledged. The solutions must be your own individual work. You are free to consult your class notes, any material posted on the class web site, and the text (Young, *An Introduction to Hilbert Space*). Please do not discuss these problems, or use other outside resources on the web or other text books. There is no time limit aside from the requirement that all solutions be submitted by 5pm on 24 October, unless you have made a previous arrangement. For clarification about procedure or problem statements, please consult the instructor.

Please complete four problems of your choosing from the six below, each worth 25 points.
(You are welcome to submit more than four, but you must designate which ones you wish to count for credit.)

1. Let c_0 denote the subspace of ℓ^∞ comprising all sequences (x_n) which tend to zero as $n \to \infty$.

   (a) Prove that c_0 is closed in ℓ^∞ with respect to $\|\cdot\|_\infty$.

   (b) Is c_0 open in ℓ^∞ with respect to $\|\cdot\|_\infty$? Prove or give a counterexample.

   (c) Is c_0 a subspace of ℓ^1? Explain.

   [cf. Young, problem 2.7]

2. Let E be a normed space. Show that E is a Banach space if and only if $\sum_{j=1}^\infty \|x_j\| < \infty$ implies that the series $\sum_{j=1}^\infty x_j$ converges in E.

   [Kirillov and Gvishiani, problem 326]

3. This problem generalizes exercise 4.x from Problem Set 4. Assume throughout that, in a Hilbert space H, (e_k) is a complete orthonormal sequence and (f_k) is an orthonormal sequence whose potential completeness we would like to determine.

   (a) Suppose that there exists $n \ge 1$ such that $\sum_{j=n+1}^\infty \|e_j - f_j\|^2 < 1$. Then show that if some $x \in H$ is orthogonal to both

   $$g_k = e_k - \sum_{j=n+1}^\infty (e_k, f_j) f_j, \qquad k = 1, \ldots, n$$

   and also to f_k for all $k > n$, then $x = 0$.

   (b) Let S denote the subspace of H orthogonal to $f_{n+1}, f_{n+2}, \ldots$. Show that $S = \mathrm{lin}\{g_1, \ldots, g_n\}$.

   (c) Now prove that if $\sum_{k=1}^\infty \|e_k - f_k\|^2 < \infty$, then the sequence (f_k) is complete.

   [Birkhoff and Rota, 1960]

4. Let H be a Hilbert space and let $A \in \mathcal{L}(H)$. Prove that $(\text{Range } A)^\perp = \text{Ker } A^*$ and that $(\text{Ker } A)^\perp$ is the closure of Range A^*. Prove also that Ker $A^*A =$ Ker A.
   This result generalizes what Strang calls the "Fundamental Theorem of Linear Algebra".

   [Young, problem 7.29]

5. Let $(e_n)_1^\infty$, $(f_m)_1^\infty$ be complete orthonormal sequences in Hilbert spaces H, K respectively and let $A \in \mathcal{L}(H,K)$. Show that

$$\sum_{n=1}^{\infty} \|Ae_n\|^2 = \sum_{m=1}^{\infty} \|A^* f_m\|^2.$$

Deduce that the quantity $\sum_{n=1}^{\infty} \|Ae_n\|^2$ has the same value for every choice of the complete orthonormal sequence (e_n) in H.

The quantity $\{\sum_{n=1}^{\infty} \|Ae_n\|^2\}^{1/2}$, if finite, is called the *Hilbert–Schmidt norm* of A. Show that it equals

$$\left\{ \sum_{i,j=1}^{\infty} |a_{ij}|^2 \right\}^{1/2}$$

if A has matrix $[a_{ij}]$ with respect to any pair of complete orthonormal sequences in H and K (i.e., $a_{ij} = (Ae_j, f_i)$).

[Young, problem 8.5]

6. This problem concerns *spectral perturbation theory*: how does the spectrum of a bounded linear operator change when that operator is modified?

Suppose H is a Hilbert space, $A \in \mathcal{L}(H)$, and $z \notin \sigma(A)$.

(a) Show that if $E \in \mathcal{L}(H)$ with $\|E\| < 1/\|(z-A)^{-1}\|$, then $z \notin \sigma(A+E)$.

(b) For any $\delta > 1/\|(z-A)^{-1}\|$, construct an operator $E \in \mathcal{L}(H)$ with $\|E\| \le \delta$ such that $z \in \sigma(A+E)$. [Hint: one can build a rank-1 E such that z is an eigenvalue of $A+E$.]

(c) Use part (b) to conclude that $\min_{w \in W(A)} |w-z| \le 1/\|(z-A)^{-1}\|$, where $W(A)$ denotes the numerical range of A: $W(A) = \{(Ax,x) : \|x\| = 1\}$.

Please write out and sign the traditional pledge.