CAAM 540 · APPLIED FUNCTIONAL ANALYSIS

Pledged Problem Set 2

Updated version posted Monday 1 December 2008. Due Friday 5 December 2008.

This problem set is to be pledged. The solutions must be your own individual work. You are free to consult your class notes and handouts, any material posted on the class web site, MATLAB/Mathematica/Maple, and the text (Young, *An Introduction to Hilbert Space*). Please do not discuss these problems, or use other outside resources on the web or other text books. There is no time limit. For clarification about procedure or problem statements, please consult the instructor.

Solve four of the following five problems. Each is worth 25 points.

1. Let H be a Hilbert space and suppose $A \in L(H)$ is *normal*, that is, $A^*A = AA^*$.

   (a) Show that $\|A^n\| = \|A\|^n$ for all positive integers n.
   (Since $\sup_{\lambda\in\sigma(A)} |\lambda| = \lim_{n\to\infty} \|A^n\|^{1/n}$, this implies that $\|A\| = \sup_{\lambda\in\sigma(A)} |\lambda|$.)

   (b) Show that $\|(z - A)^{-1}\| = \left(\inf_{\lambda\in\sigma(A)} |z - \lambda|\right)^{-1}$.

   (c) What does part (b) imply about the sensitivity of the spectrum to small perturbations of A, that is, what can be said of $\sigma(A+E)$ for small $\|E\|$? (You may use the results of Problem 6 from the first pledged problem set.)

2. Let A be an (possibly unbounded) operator on a Hilbert space H, and suppose that for some $z \in \mathbb{C}$, the resolvent $(z - A)^{-1}$ exists and is compact.

   (a) Show that if y is any other point in the resolvent set, then $(y - A)^{-1}$ is also compact.
   (Hint: you may use Problem 8.1 from Problem Set 7.)

   (b) What can be deduced about the spectrum of A and $\mathrm{Ker}(\lambda - A)$ for any $\lambda \in \sigma(A)$?

   (c) Show that if A is bounded and has compact resolvent, then $\mathrm{Ran}(A)$ must be finite dimensional.

3. Consider the multiplication operator $A : L^2[0, 1] \to L^2[0, 1]$ defined by

$$Af(x) = xf(x).$$

   (a) Show that $\sigma(A) = \sigma_c(A) = [0, 1]$, where $\sigma_c(\cdot)$ denotes the continuous spectrum, i.e., those points $z \in \mathbb{C}$ for which $(z - A)^{-1}$ exists and is densely defined, but is unbounded.

   (b) Determine the spectral family $(E_\lambda)_{\lambda\in\mathbb{R}}$ for this operator.

[adapted from Kreyszig]

4. This question concerns banded Laurent operators (studied in Problem 7.z on Problem Set 6) on $\ell^2(\mathbb{Z})$ and the related *banded Toeplitz operators* on $\ell^2(\mathbb{N})$. Given a *symbol function*

$$a(z) = \sum_{k=-m}^{n} a_k z^k,$$

here we will denote the associated doubly-infinite Laurent operator as $L(a)$, and define the *singly-infinite* Toeplitz operator, $T(a)$, to be the restriction of $L(a)$ to its 'bottom right corner'; more precisely, $T(a)$ is the operator on $\ell^2(\mathbb{N})$ whose matrix representation has constant values a_ℓ on the ℓth diagonal for $\ell = -m, \ldots, n$, and zeros everywhere else. For example, with $m = 1$ and $n = 2$, we would have

$$T(a) = \begin{bmatrix} a_0 & a_1 & a_2 & & & \\ a_{-1} & a_0 & a_1 & a_2 & & \\ & a_{-1} & a_0 & a_1 & a_2 & \\ & & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

with the unspecified entries equation to zero.

(a) The Laurent operator $L(a)$ is normal, i.e., $L(a)^*L(a) = L(a)L(a)^*$. (You don't need to prove this, but you can if you like.) Use Problem 1 on this set to determine $\|L(a)\|$. (You may use the fact that the spectrum of $L(a)$ acting on $\ell^2(\mathbb{Z})$ is the same as the spectrum of $L(a)$ acting on $\ell^\infty(\mathbb{Z})$.)

(b) Show that $\|T(a)\|_{\ell^2(\mathbb{N})} \le \|L(a)\|_{\ell^2(\mathbb{Z})}$.

(c) Suppose the symbol is upper triangular, i.e., $m = 0$ and $n \ge 0$. What is the spectrum of $T(a)$?

(d) Sketch/plot the spectrum $\sigma(T(a))$ for $a(z) = z + z^2$.

5. This is a computational continuation of the previous problem. Consider the upper-triangular Toeplitz operator on $\ell_2(\mathbb{N})$ with symbol $a(z) = 4z + 3z^4$.

Let $T_N(a) \in \mathbb{C}^{N\times N}$ denote the $N \times N$ finite section of A; this will be a Toeplitz matrix that is zero everywhere except for 4 on the first superdiagonal and 3 on the fourth superdiagonal. (You *should not* add any extra entries in the corners, as one would need to do when taking finite sections of the Laurent operator.)

(a) How does the spectrum of $T_N(a)$ compare to the spectrum of $T(a)$?

(b) Produce a plot showing $\|(zI - A_N)^{-1}\|$ for $N = 2^k$ for $k = 2, \ldots, 8$ for each of $z = -7, -3, 3, 7$. (Use logarithmic scales for the horizontal and vertical axes.) How does the rate at which the norm of the resolvent grows with N relate to the symbol curve $\{a(\mathrm{e}^{\mathrm{i}\theta}) : \theta \in [0, 2\pi)\}$?

(c) Produce a contour plot showing $\|(zI - A_{32})^{-1}\|$ on a grid in the complex plane over $\mathrm{Re}\, z \in [-8, 8]$ and $\mathrm{Im}\, z \in [-8, 8]$. Show contours for $\|(zI - A_{32})^{-1}\| = 10^0, 10^2, 10^6$. Interpret your results in light of Problem 6 on the first pledged problem set. How does your contour plot compare to the symbol curve $\{a(\mathrm{e}^{\mathrm{i}\theta}) : \theta \in [0, 2\pi)\}$?

Please write out and sign the traditional pledge.