Matrix - Vector Product

$A \in \mathbb{C}^{m \times n}$, $x \in \mathbb{C}^n$, $b \in \mathbb{C}^m$

$b = Ax$

$b_i = \sum_{j=1}^{n} a_{ij} x_j$ component wise definition

$Ax = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$ Linear Combo. of Columns

where $A = [a_1, a_2, \cdots, a_n]$ a_j - j^{th} column of A

Adjoint or Hermitian-Conjugate (Conjugate-Transpose)

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$, $A^* = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{21} & \bar{a}_{31} \\ \bar{a}_{12} & \bar{a}_{22} & \bar{a}_{32} \end{bmatrix}$

If $A \in \mathbb{R}^{m \times n}$ $A^* = A^T$ (transpose)

Can always write $A = C + iB$; $B, C \in \mathbb{R}^{m \times n}$

$A^* = C^T - iB^T$

Inner Product $x, y \in \mathbb{C}^n$

$$x^* y = \sum_{j=1}^{n} \bar{x}_j y_j$$

Matrix - Matrix Product

$C = AB \qquad A \in \mathbb{C}^{m\times k},\ B \in \mathbb{C}^{k\times n},\ C \in \mathbb{C}^{m\times n}$

k - Must be the same for this to be defined

Columnwise Definition

$$C = [c_1, c_2, \dots, c_n] = [Ab_1, Ab_2, \dots, Ab_n]$$

Row Dfn

$$C = \begin{bmatrix} c_1^* \\ c_2^* \\ \vdots \\ c_m^* \end{bmatrix} = \begin{bmatrix} a_1^* B \\ a_2^* B \\ \vdots \\ a_m^* B \end{bmatrix}$$

where $\begin{bmatrix} a_1^* \\ a_2^* \\ \vdots \\ a_m^* \end{bmatrix} = A$, a_j^* is j-th row of A

Outer Product Dfn

$$A = [a_1, a_2, \dots, a_k], \quad B = \begin{bmatrix} b_1^* \\ b_2^* \\ \vdots \\ b_k^* \end{bmatrix}$$

$$AB = a_1 b_1^* + a_2 b_2^* + \cdots + a_k b_k^*$$

Inner Product Dfn

$$A = \begin{bmatrix} a_1^* \\ a_2^* \\ \vdots \\ a_m^* \end{bmatrix}, \quad B = [b_1, b_2, \dots, b_n]$$

$$c_{ij} = a_i^* b_j$$

Thm: $(AB)^* = B^* A^*$ Fact $AB \neq BA$ in general (even when defined)

Dfn Range $(A) = \{Ax : x \in \mathbb{C}^n\}$ (Column-Space)

Dfn Null $(A) = \{x \in \mathbb{C}^n : Ax = 0\}$ ~~Nullity~~ (Null Space)

Exercise: A subset S of a given vector space $\mathbb{X}$ is a <u>subspace</u> if $S \subset \mathbb{X}$ and S is a vector space.

Show Range(A) is a subspace of $\mathbb{C}^m$ and Null(A) is a subspace of $\mathbb{C}^n$ (Reminder: It is sufficient to show

$$x, y \in S \Rightarrow x + y \in S$$
$$x \in S \Rightarrow x\alpha \in S \quad \text{any scalar } \alpha)$$

Dfn ~~Thm~~ Rank(A) = dim Range(A) dim - dimension

= # linearly independent vectors in a basis

(Review: If you have forgotten, remind yourself of the definitions of span, linearly independent, and basis)

Thm Rank(A) = Rank(A^*)

Thm: $A \in \mathbb{C}^{m \times n}$, $m \geq n$ has <u>full rank</u> ($\mathrm{Rank}(A) = n$) iff $Ax = 0 \Rightarrow x = 0$

Thm: A is nonsingular (A^{-1} exists) iff A is square ($n \times n$) with full rank n ($A \in \mathbb{C}^{n \times n}$, $\mathrm{Rank}(A) = n$)

Pf: (sketch)

($\Rightarrow$) If A^{-1} exists $\quad Ax = 0 \Rightarrow A^{-1}Ax = A^{-1}0 \Rightarrow x = 0$.

($\Leftarrow$) If A is $n \times n$ and $\mathrm{Rank}(A) = n$ then the columns of A are linearly independent (that is the meaning of $Ax = 0 \Rightarrow x = 0$) Since there are n columns of A, they ~~span $\mathbb{C}^n$ &~~ are a basis for $\mathbb{C}^n$. Thus $Ax_j = e_j$ has a unique solution x_j for each unit vector $e_j^* = [0, \cdots 0, 1, 0, \cdots 0]$

$$\underset{j\text{-th}}{}$$

Hence

$$A X = I \quad , \; X = [x_1, x_2, \cdots, x_n] \quad , \; X = A^{-1}$$

When we write $x = A^{-1}b$ we really mean x is the unique vector of coefficients such that

$$Ax = b$$

The components of x give the representation of b in the column basis

Polynomial Interpolation

$$\mathcal{P}_n = \{ p : p \text{ is a polynomial of degree less than } n \}$$

Exercise: Check that $\mathcal{P}_n$ is a vector space over the field $\mathbb{C}$ and that the monomials $\{1, t, t^2, \ldots, t^{n-1}\}$ form a basis for $\mathcal{P}_n$. What is the dimension of $\mathcal{P}_n$?

Any $p \in \mathcal{P}_n$ can be expressed as

$$p(t) = 1 \cdot c_0 + t c_1 + t^2 c_2 + \cdots + t^{n-1} c_{n-1}$$

$p(t)$ is a function ("a vector with infinite no. components")

Thus we can identify $\mathcal{P}_n$ with $\mathbb{C}^n$

$$p(t) \iff c = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

$$p(t) = \quad = [1, t, t^2, \ldots, t^{n-1}] \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

Is this identification unique?
ie Is it true that there is one and only one vector $c \in \mathbb{C}^n$ associated with each polynomial $p \in \mathcal{P}_n$

Yes: If
$$p(t) = c_0 + t c_1 + t^2 c_2 + \cdots + t^{n-1} c_n$$
$$= \hat{c}_0 + t \hat{c}_1 + t^2 \hat{c}_2 + \cdots + t^{n-1} \hat{c}_n$$

Then $0 = (c_0 - \hat{c}_0) + t(c_1 - \hat{c}_1) + \cdots + t^{n-1}(c_n - \hat{c}_n)$ is true for every t. But a nonzero polynomial of degree $n-1$ has only $n-1$ roots so $c_j = \hat{c}_j \quad 0 \le j \le n-1$.

Dfn: Linear Transformation
A function (map) $T : X \to Y$ from a vector space X to a vector space Y is linear if

$$T(x+z) = T(x) + T(z), \quad \text{all } x, z \in X$$
$$T(x\alpha) = T(x)\alpha \quad \text{for any scalar } \alpha$$

Matrix Representation of a linear transformation

Let $\{x_1, \dots, x_n\}$ be a basis for $\mathcal{X}$

Then $A = [T(x_1), T(x_2), \dots, T(x_n)]$
is a matrix representation of T

Exercise: Verify This

Consider the map T from $\mathcal{P}_n$ into $\mathbb{C}^m$
defined by
$$p \xrightarrow{T} \begin{bmatrix} p(t_1) \\ p(t_2) \\ \vdots \\ p(t_m) \end{bmatrix}$$

where $\{t_j\}$ is a set of m (specified) complex numbers.

T is linear since

$$(p+q)(t_j) = p(t_j) + q(t_j) \quad \text{all } p,q \in \mathcal{P}_n$$

$$\underbrace{(\alpha p)}_{\alpha \text{ times polynomial } p}(t_j) = \underbrace{\alpha\,(p(t_j))}_{\alpha \text{ times the } \underline{\text{value}}\ p(t_j)} \quad \text{any } \alpha \in \mathbb{C}$$

What is the matrix of this transformation?

Basis set $1, t, t^2, \dots, t^{n-1}$

$$A = \left[T(1), T(t), T(t^2), \dots, T(t^{n-1})\right]$$

$$= \begin{bmatrix} 1 & t_1 & t_1^2 & & t_1^{n-1} \\ 1 & t_2 & t_2^2 & & t_2^{n-1} \\ 1 & t_3 & t_3^2 & \cdots & t_3^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & t_m & t_m^2 & & t_m^{n-1} \end{bmatrix}$$

Vandermonde Matrix

Check: Given $p(t) = c_0 + tc_1 + t^2c_2 + \dots + t^{n-1}c_{n-1}$

$$\begin{bmatrix} p(t_1) \\ p(t_2) \\ \vdots \\ p(t_m) \end{bmatrix} = Ac \quad , \quad c = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

Polynomial Interpolation: Given a set of n values $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

Find $p \in P_n$

i.e. Find a set of coefficients $\{c_j\}$ s.t. $\begin{bmatrix} p(t_1) \\ p(t_2) \\ \vdots \\ p(t_n) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

$q_0 \dots q_{n-1}$

Polynomial Interpolation - Matrix · Vector form

Solve $\quad Ac = b$

When does A^{-1} exist (remember we ~~only~~ have exactly n points t_j)

$A \in \mathbb{C}^{n \times n}$

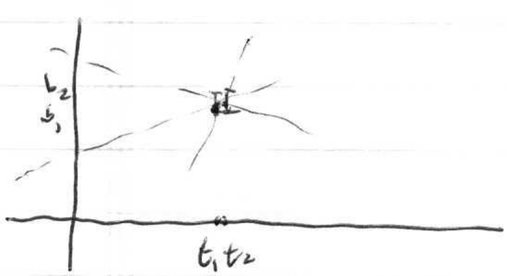
A^{-1} exists iff $\quad Ac = 0 \Rightarrow c = 0$

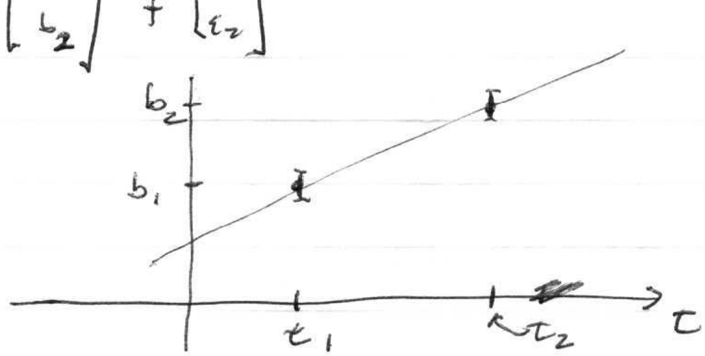
Exercise: Show this is true (A^{-1} exists) iff the $\{t_j\}$ are all distinct
(Hint: $p \in \mathcal{P}_n$ has ~~at most~~ $n-1$ roots)

What happens when $t_i \to t_j$ for some i,j?

Consider linear interpolation with perturbations

$$\begin{bmatrix} 1 & t_1 \\ 1 & t_2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$$

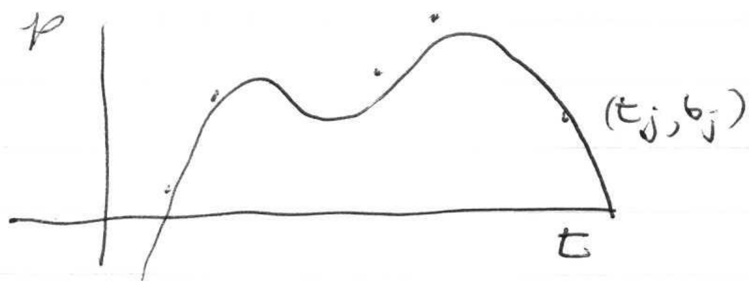




Best Polynomial Fit

Given $m \geq n$ data points (t_j, b_j)

Find a polynomial $p \in \mathcal{P}_n$ that "best fits" the data



"Solve" $Ac = b$

Problem: $b \notin \text{Range}(A)$

Best Fit: put $z = b - Ac$

Let z_i be ith component

$$z_i = b_i - [1, t_i, t_i^2, \dots, t_i^{n-1}] \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

Adjust c to solve

$$\min_c \sum_{i=1}^{m} |z_i^2| = \min_c \sum_{i=1} \bar{z}_i z_i$$

$$= \min_c z^* z$$

$$= \min_c (b-Ac)^*(b-Ac)$$

Euclidean Length (2-norm)

$$\|z\|_2 = \sqrt{z^* z}$$

Dfn: The "angle" θ between vectors $x, y \in \mathbb{C}^n$ is the (smallest positive θ) st.

$$\cos(\theta) = \frac{x^* y}{\|x\|_2 \|y\|_2}$$

Exercise: Show this definition has the "right" interpretation when $x, y \in \mathbb{R}^2$

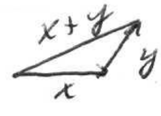
Properties of Inner Product

$$\left.\begin{aligned} (x_1 + x_2)^* y &= x_1^* y + x_2^* y \\ x^*(y_1 + y_2) &= x^* y_1 + x^* y_2 \end{aligned}\right\} \text{bi-linear}$$

$$(\alpha x)^*(\beta y) = \bar{\alpha}\beta\, x^* y$$

Properties of $\|\cdot\|_2$

1) $z \neq 0 \Rightarrow \|z\|_2 > 0$
2) $\|x + y\|_2 \leq \|x\|_2 + \|y\|_2$



3) $\|x\alpha\|_2 = |\alpha|\,\|x\|_2$

Orthogonality

Dfn x, y are orthogonal if $x^* y = 0$

(They are at right angles to each other is in $\mathbb{R}^n$)

Dfn: A set $S = \{q_1, \dots, q_k\}$ of non zero vectors is an orthogonal set if

$$q_i^* q_j = 0 \quad i \neq j$$

S is an orthonormal set if in addition

$$q_i^* q_i = 1$$

Exercise: How can you turn any orthogonal set S into an orthonormal set?

Thm The vectors in an orthogonal set $S = \{q_1 \cdots q_k\}$ are linearly independent

Proof: w.l.o.g. assume S is orthonormal

Suppose $\quad q_1 c_1 + q_2 c_2 + \cdots + q_k c_k = 0$

For any j: $\quad c_j = q_j^* q_1 c_1 + \cdots + q_j^* q_j c_j + \cdots + q_j^* q_k c_k = 0$

$\boxtimes$

Dfn: A matrix $Q \in \mathbb{C}^{m \times m}$ is <u>unitary</u> if $Q^* Q = I_m$

(ie $Q^* = Q^{-1}$)

Note: If Q is real we generally say it is orthogonal ie $Q^T Q = I_m$

Fact: A matrix $Q = [q_1, \cdots q_m] \in \mathbb{C}^{m \times n}$

is unitary iff $\{q_1, \cdots q_m\}$ is an <u>orthonormal</u> set

Fact: Since a left inverse is also a right inverse we have $QQ^* = Q^* Q = I$

Given any $v \in \mathbb{C}^m$ we can write

$$v = Iv = QQ^* v$$

The vector $\hat{v} = Q^* v$ is the vector of coefficients representing v in the ortho-normal bases $\{q_1 \cdots q_m\}$

Two interpretations of

$$v = QQ^*v$$

1) $$QQ^*v = [q_1, q_2, \cdots, q_m] \begin{bmatrix} q_1^* v \\ q_2^* v \\ \vdots \\ q_m^* v \end{bmatrix}$$

$$= \sum q_j \, (\underbrace{q_j^* v}_{\text{expansion coefficients}})$$

2) $$QQ^*v = [q_1 q_1^* + q_2 q_2^* + \cdots + q_m q_m^*] v$$

$$= (q_1 q_1^*)v + (q_2 q_2^*)v + \cdots + (q_m q_m^*)v$$

(Sequence of 1-Dimensional "Projections")