Problem 1

Pick a nonsymmetric linear system from Matrix Market or from the Tim Davis collection and solve it using the code in compLinSlvrs.m from the CAAM 551 web page. Pick a problem with \( n \geq 1000 \) from an application area that is of interest to you.

I would like to see the graph comparing the convergence history of the methods and also a comparison of computing times (use tic toc commands).

I would like you to do this for the original matrix and again for the pre-conditioned problem using a pre-conditioner of your choice.

Problem 2

Do problem 4 page 212 of Saad and write a Matlab code to implement the method (please demonstrate that the code works on a test problem of your choice).

Problem 2

Let \( AV = VH + fe_k^T \) be a \( k \)-step Arnoldi factorization of \( A \) with \( Ve_1 = b \) (assume \( \|b\| = 1 \) where \( \| \cdot \| \) is the 2-norm). Assume \( A \) is nonsingular.

Let

\[ r_k = p_k(A)b \]

be the GMRES residual (i.e. \( r_k = b - Ax_k \), where \( x_k \) is computed with the GMRES method).

Prove the following results:

1. The \( j + 1 \)-st column \( v_{j+1} = Ve_{j+1} \) of \( V \) is of the form \( \phi_j(A)b \) where \( \phi_j(\tau) = \gamma_j \det(\tau I - H_j) \) with \( H_j \) the leading \( j \times j \) submatrix of \( H \) (i.e. the value of \( H \) at step \( j \) of the Arnoldi process).
2. The roots $\theta_j$ of the GMRES polynomial $p_k(\tau)$ are the eigenvalues of the generalized eigenvalue problem

$$\overline{H}^T \overline{H} z = H^T z \theta,$$

where $\overline{H}^T = [H^T, e_k \beta]$ with $\beta = \|f\|$.

These are called Harmonic Ritz values.

3. When $A$ is symmetric, demonstrate that these Harmonic Ritz values are reciprocals of the critical points of the generalized Rayleigh quotients

$$\frac{w^T A^{-1} w}{w^T w}, \text{ where } w \in A\mathcal{K}_k(A, b).$$

4. Let $A$ be symmetric and indefinite with $\lambda_-$ the (algebraically) largest negative eigenvalue and $\lambda_+$ the smallest positive eigenvalue of $A$. Prove there are no Harmonic Ritz values in the open interval $(\lambda_-, \lambda_+)$. 

5. If $x_k \in \mathcal{K}_k(A, b)$ is any approximate solution drawn from the Krylov space (not necessarily the GMRES approximation) then $b - Ax_k = p(A)b$ for some polynomial $p$ of degree $k$ such that $p(0) = 1$ and where $x_k = \phi(A)b$ with

$$p(\tau) = 1 - \tau \phi(\tau).$$

Show that $\phi$ is the unique polynomial of degree $k - 1$ that interpolates the function $\eta(\tau) \equiv \frac{1}{\tau}$ at the $k$ roots of $p$.

6. Graph the polynomial $\phi$ from GMRES with $k = 20$ for $A$ as the negative of the 1-D discrete Laplacian of order 100 ($A = \text{trid}[-1, 2, -1]$). Your graph should go over the interval $(\lambda_1, \lambda_n)$, the eigenvalue range of $A$. On the same plot, graph the function $\eta(\tau) = \frac{1}{\tau}$ over the same interval and show the interpolation points. Repeat this for $A = 2I$ in place of $A$. 

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