CAAM 551 Problem set 4

Due Fri. 30 Oct. 09

Problem 1

Run the Matlab code `powersR.m` with $\rho = .1$ and $\tau = 10$ (i.e. invoke `powersR(.1,10)`). Explain the convergence behavior shown in the plots. In particular, explain why the curves only start behaving as predicted by the spectral radius ρ sometime after the iteration number is greater than 100.

Problem 2

Let $\|\cdot\|$ denote any vector norm on $\mathbf{R}^n$ and also to denote the corresponding induced matrix norm on $\mathbf{R}^{n\times n}$.

1. Let B be any nonsingular $n \times n$ matrix. Let $\|\cdot\|_B$ be defined by

   $$\|x\|_B = \|Bx\|$$

   for any $x \in \mathbf{R}^n$. Prove that $\|\cdot\|_B$ is a vector norm on $\mathbf{R}^n$.

2. Let $\|\cdot\|_B$ also denote the matrix norm induced by this vector norm. Show

   $$\|A\|_B = \|BAB^{-1}\|.$$

   Note: Since $\|\cdot\|_B$ is an induced norm, it automatically satisfies all of the required multiplicitive consistency properties required by the proofs we have recently done (call this a consistent matrix norm).

3. Given any square matrix A and any positive number ϵ, explain how to construct a consistent matrix norm $\nu(\cdot)$ such that

   $$\nu(A) < \rho(A) + \epsilon,$$

   where $\rho(A)$ is the spectral radius of A. Hint: Use the Schur decomposition and construct a diagonal scaling matrix D such that the strict upper triangular part of DRD^{-1} is arbitrarily small.

4. Use the previous result (item 3) to prove that $\lim_{k\to\infty} \|A^k\|^{\frac{1}{k}} = \rho(A)$.

Problem 3

Let A be an n by n complex matrix. Let $A = QRQ^*$ be a Schur decomposition of A with $I = Q^*Q$ and R upper triangular.

1. Let p and q be polynomials. Give a necessary and sufficient condition on q involving the eigenvalues of A for the existence of $q(A)^{-1}$. Let $r(t) = p(t)/q(t)$. Derive a Schur decomposition for $p(A)$ and also a Schur decomposition for $r(A)$ (when it exists). What are the eigenvalues of $r(A)$?

2. Let $f(z) = \sum_{j=0}^{\infty} \gamma_j z^j$ be convergent in a disc $\mathcal{D}$ of radius ρ centered at the origin. Give a condition on the eigenvalues of A so that $f(A)$ is well defined. You must show that the sequence of matrix polynomials $F_k = f_k(A) = \sum_{j=0}^{k} \gamma_j A^j$ is convergent under your condition. Assuming convergence, give a Schur decomposition for $f(A)$. What are the eigenvalues of $f(A)$?

   Note: A sequence of $n \times n$ matrices $\{F_k\}$ converges to an $n \times n$ matrix F if and only if $lim_{k\to\infty} \|F_k - F\| = 0$.

Problem 4

Pick a nonsymmetric linear system from Matrix Market and solve it using the code in compLinSlvrs.m from the CAAM 551 web page. Pick a problem with $n \geq 500$ from an application area that is of interest to you.

I would like to see the graph comparing the convergence history of the methods and also a comparison of computing times (use `tic toc` commands).

I would like you to do this for the original matrix and again for the pre-conditioned problem using a pre-conditioner of your choice.

Problem 5

Let $AV = VH + fe_k^T$ be a k-step Arnoldi factorizatin of A with $Ve_1 = b$ (assume $\|b\| = 1$ where $\|\cdot\|$ is the 2-norm). Assume A is nonsingular.

Let

$r_k = p_k(A)b$ be the GMRES residual (i.e. $r_k = b - Ax_k$, where x_k is computed with the GMRES method).

Prove the following results:

1. The $j+1$-st column $v_{j+1} = Ve_{j+1}$ of V is of the form $\phi_j(A)b$ where $\phi_j(\tau) = \gamma_j \det(\tau I - H_j)$ with H_j the leading $j \times j$ submatrix of H (i.e. the value of H at step j of the Arnoldi process). Show that the roots of the FOM polynomial at step j are the eigenvalues of H_j These are called *Ritz values.*

2. Show that the roots θ_j of the GMRES polynomial $p_k(\tau)$ are the eigenvalues of the generalized eigenvalue problem

   $$\overline{H}^T \overline{H} z = H^T z \theta,$$

   where $\overline{H}^T = [H^T, e_k \beta]$ with $\beta = \|f\|$.

   These are called *Harmonic Ritz values.*

Problem 6

Do problem 4 page 212 of Saad and write a Matlab code to implement the method (please demonstrate that the code works on a test problem of your choice).