(1) (25 points)
(a) (5 points) Let \( g \in C^2([a, b]) \). Show that if \( \max_{x \in (a, b)} |g'(x)| < 1 \), \( x = g(x) \) is a contraction.
(b) (10 points) Assume that \( f(x) \in C^2(\mathbb{R}) \) and \( f'(x) \neq 0 \). Re-write Newton’s method as a fixed point iteration and show that if it converges, the convergence rate is quadratic. Derive a condition involving \( f(x), f'(x), f''(x) \) under which Newton’s method is a contraction.
(c) (10 points) Let \( f(x) = x^2 - a \) for \( a > 0 \). For what values of \( x \) will Newton’s method be a contraction?

(2) (25 points) Consider the evaluation of a smooth function \( f(x) \) on 5 equally spaced grid points \( x_{-2}, x_{-1}, x_0, x_1, x_2 \) with \( x_{i+1} - x_i = h \).
(a) (10 points) Show that the error for the following central difference approximation
\[
\frac{\partial f(x)}{\partial x} \approx \frac{f(x + h) - f(x - h)}{2h}
\]
is \( O(h^2) \), and determine an \( O(h^4) \) accurate finite difference approximation for \( \frac{\partial f(x_0)}{\partial x} \) using the function values \( f(x_{-2}), \ldots, f(x_2) \) (you do not need to use every function evaluation).
(b) (15 points) Suppose you now have function evaluations at points \( x_{-k}, \ldots, x_k \). Describe how to construct an \( O(h^{2k}) \) approximation for \( \frac{\partial f(x_0)}{\partial x} \). You do not have to give an explicit formula, but be as descriptive as possible.

(3) (25 points) Let \( f(x) \) be in \( C^{2n+2}([-1, 1]) \).
(a) (5 points) Let \( p(x) \) be the Hermite interpolant at the points \( x_1, \ldots, x_n \in [-1, 1] \). State the error formula for \( |f(x) - p(x)| \). Explain what interpolation points one should use for Hermite interpolation, and briefly explain why.
(b) (15 points) Let \( x_1 = -1, \ldots, x_n = 1 \) be equally spaced points with spacing \( h \). Assume the values of \( f(x_i), f'(x_i) \) at points \( x_1, \ldots, x_n \) are known. Derive an error bound for piecewise cubic Hermite interpolation. For \( f(x) = \sin(x) \) and 5 points, can you guarantee that the error is below \( 10^{-5} \)?
(c) (5 points) Suggest a way to construct an approximate piecewise cubic Hermite interpolant if the values of \( f(x_i) \) are known but the values of \( f'(x_i) \) are not.

(4) (25 points) The Chebyshev polynomials are defined through \( T_n(x) = \cos(n \cos^{-1}(x)) \).
(a) (10 points) Show that \( T_n(x) \) satisfies the properties
\[
T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)
\]
\[
2T_m(x)T_n(x) = T_{m+n}(x) + T_{|m-n|}(x)
\]
and determine explicit expressions for \( T_n(-1), T_n(1), \) and \( T_n(0) \). Hint: use the trigonometric identities
\[
\cos(a \pm b) = \cos(a) \cos(b) \mp \sin(a) \sin(b)
\]
\[
2 \cos(a) \cos(b) = \cos(a + b) + \cos(a - b).
\]
(b) (15 points) Determine explicitly the nodes and weights of the 2-point Gaussian quadrature formula for the integral
\[
\int_{-1}^{1} \frac{f(x)}{\sqrt{1 - x^2}} \, dx.
\]
What is the set of functions that can be integrated exactly by this quadrature rule?