(1) (25 points) For $\theta \in [0, 1]$, consider the scheme

$$u_{n+1} = u_n + hf(t_n + (1 - \theta)h, \theta u_n + h(1 - \theta)u_{n+1})$$

for solving the ODE $u' = f(t, u)$.

(a) For all $\theta \in [0, 1]$, find the order of the scheme. Hint: you may wish to do so by determining the largest $k$ such that the scheme is exact for monomial solutions $u(t) = t^k$.

(b) For $\theta = 0$ and $1$, determine the stability domain of the scheme.

(c) If you were to solve the ODE

$$u' = \begin{bmatrix} -1000 & 1 \\ 0 & -1000 \end{bmatrix} u$$

using this scheme until $u(t)$ is reasonably close to equilibrium, is $\theta = 0$ suitable? Is $\theta = 1$ suitable?

(2) (25 points) Let $A$ be an $n \times n$ matrix.

(a) Let $A$ be a triangular matrix with distinct diagonal entries. Prove that the diagonal entries of $A$ are eigenvalues, and that the matrix of eigenvectors is also triangular. Describe how to find an eigenvector (up to scaling) corresponding to a given diagonal entry. You may use without proof that a triangular matrix is invertible if and only if the diagonal entries are non-zero.

(b) $A$ is skew-symmetric if $A = -A^*$. Show that skew symmetric matrices have purely imaginary eigenvalues, and that eigenvectors corresponding to distinct eigenvalues are orthogonal.

(c) Write down expressions for the eigenvalues and eigenvectors of the skew-symmetric matrix

$$\begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix}$$

in terms of singular values and singular vectors of $A$.

(3) (25 points) Let $A$ be a $n \times n$ real matrix and let $b$ be a $n \times 1$ vector. Consider the minimization problem

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|^2 + \lambda \|x\|^2.$$

Let $x_\lambda$ denote the solution.

(a) Characterize the solution by deriving an equivalent linear system.

(b) Show that, for $\lambda > 0$ the solution always exists.

(c) Characterize the solution in terms of the singular values and singular vectors of $A$.

(d) Suppose rank of $A$ is less than $n$. Describe $\lim_{\lambda \to 0} x_\lambda$.

(4) (a) Consider the multistep method

$$x_{n+2} + 2x_{n+1} - 3x_n = \frac{h}{10} (f_{n+2} + 16f_{n+1} + 17f_n).$$

Is this method convergent?

(b) Consider the two step method

$$y_{n+2} = y_{n+1} + \frac{h}{2} (3f_{n+1} - f_n).$$

Is this method convergent?

(c) Find the order of the methods in parts (a) and (b).

Make sure to state any relevant theorems for each part.